PERIYAR UNIVERSITY

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SALEM - 636 011

CENTRE FOR DISTANCE AND ONLINE EDUCATION

(CDOE)

BACHELOR OF BUSINESS ADMINISTRATION

SEMESTER - I

CORE – I: OPERATION RESEARCH

(Candidates admitted from 2024 onwards)

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UNIT I

UNIT OBJECTIVE

Leaner programming principles focus on eliminating unnecessary steps, optimizing resource utilization, and continuously improving workflows through iterative cycles. By adopting lean practices, teams aim to enhance productivity, shorten development cycles, and enhance overall software quality. This approach not only benefits organizations by reducing costs and increasing competitiveness but also fosters a culture of innovation and responsiveness to customer needs. Ultimately, studying leaner programming aims to cultivate a more agile and effective software development environment capable of rapidly adapting to changing market demands.

1.1 CONCEPT AND SCOPE OF OPERATIONS RESEARCH

Origin: Operations Research came into existence and gained prominence during the World War II in Britain with the establishment of team of scientists to study the strategic and tactical problems of various military operations. Scientists of different disciplines were part of this team, their research on military operation soon find applications in other fields also. Now, it was started applying in the fields of industry, trade, agriculture, planning and various other fields of economy and named as 'Operations Research'. Hence the scientific methods and techniques of Operations Research became equally useful for the planners, economists, administrators, irrigation or agricultural experts and statisticians etc. The use of Operations Research has not limited to the Britain only. Many countries of the world had started using O.R. India was one of the few first countries who started using O. R. Regional Research Laboratory located at Hyderabad was the first Operations Research unit established in India during 1949. With the opening of this unit Operations Research in India came into existence. At the same time one more unit was set up in Defence Science Laboratory. In 1955, Operations Research Society of India was formed. Today, O.R. became a professional discipline and studied as a popular subject in Management institutes and school of Mathematics.

1.1.2 Definitions

Operations Research can be defined simply as combination of two words operation and research where operation means some action applied in any area of interest and research imply some organized process of getting and analysing information about the problem environment.

However, many scientists or experts has been defined O.R. in various ways but the opinions about the definitions of it have been changed according to the growth of the subject. So before defining O.R. it is important to see few definitions of it.

1. O.R. is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.-Morse and Kimbal (1946).
2. O.R. is a scientific method of providing executive with an analytical and objective basis for decisions.-P.M.S. Blackett (1948).
3. O.R. is the application of scientific methods, techniques and tools to problems involving the operations of system so as to provide these in control of the operations with optimum solutions to the problem.-Churchman, Acoff, Arnoff (1957).
4. O.R. is a management activity pursued in two complementary ways one-half by the free and bold exercise of commonsense untrammeled by any routine, and other half by the application of a repertoire of well-established pre created methods and techniques.-Jagjit Singh (1968).

On the basis of all above opinions, Operations Research can be defined in more general and comprehensive way as:

"Operation research is a branch of science which is concerned with the application of scientific methods and techniques to decision making problems and with establishing the optimal solutions".

1.1.2 Scope of operations research

Scope of O.R. is very wide in today's world as it provides better solution to various decision-making problems with great speed and efficiency. Areas where methods/models developed in Operations Research can be applied are given here under:

In Agriculture

With the explosion of population and consequent shortage of food, every country is facing the problem of optimum allocation of land to various crops in accordance with the climatic conditions, optimum distribution of water from different resources. Problems of agriculture production under various restrictions can be solved by applications of Operations Research techniques.

In Defence Operations

Since Second World War operation research have been used for Defence operations with the aim of obtaining maximum gains with minimum efforts.

In Finance

In these modern times, government of every country or every organisation wants to introduce such type of planning/policies regarding their finance and accounting which optimize capital investment, determine optimal replacement strategies, apply cash flow analysis for long range capital investments, formulate credit policies, credit risk. Techniques developed in O.R. can be applied for attaining above said things.

In Marketing

A Marketing Administrator has to face many problems like production selection, formulation of competitive strategies, distribution strategies, selection of advertising media with respect to cost and time, finding the optimal number of salesmen, finding optimum time to launch a product. All such problems can be overcome using Operations Research Techniques.

In Personnel Management

Every organization wants to make selection of personnel on minimum salary. It needs to find the best combination of workers in different categories with respect to costs, skills, age and nature of jobs. It also needs to frame recruitment policies, assign jobs to machines or workers.

In LIC

Operations Research Techniques can be fruitfully applied in LIC offices as it enables the policy makers to decide the premium rates for various modes of policies.

In Research and Development

In determination of the areas of concentration of research and development. It also helps in project selection.

O.R. helps in solving many other problems faced by public as well as private sectors such as the ones in economic and social planning, management of natural resources, energy, housing pollution control, waiting lines and administrative problems, insurance policies and many more.

Let us Sum Up

The fundamentals of linear programming and its scope and definition helps the students to understand the concepts better

1.1 Check Your Progress

1. What is the primary goal of leaner programming?

- A) Maximizing complexity in software development
- B) Minimizing customer feedback
- C) Enhancing team productivity and efficiency
- D) Increasing software bugs

2. Leaner programming aims to achieve which of the following?

- A) Long development cycles
- B) High levels of waste in processes
- C) Continuous improvement and waste reduction
- D) Isolation from customer feedback

3. Which principle is NOT a core aspect of leaner programming?

- A) Eliminating unnecessary steps
- B) Maximizing value delivery
- C) Increasing bureaucracy
- D) Optimizing resource utilization

4. What does leaner programming emphasize in software development?

- A) Complexity over simplicity
- B) Rapid iteration and feedback
- C) Lengthy documentation
- D) Single-function teams

5. How does leaner programming benefit organizations?

- A) By slowing down development processes
- B) By reducing costs and increasing competitiveness
- C) By decreasing team collaboration
- D) By ignoring customer needs

1.2 LINEAR PROGRAMMING PROBLEMS -AN INTRODUCTION

A mathematical programming is an optimization technique by which the maximum or minimum value of a function is determined under certain conditions. Mathematical programming in which constraints are expressed as linear equalities / inequalities is called linear programming.

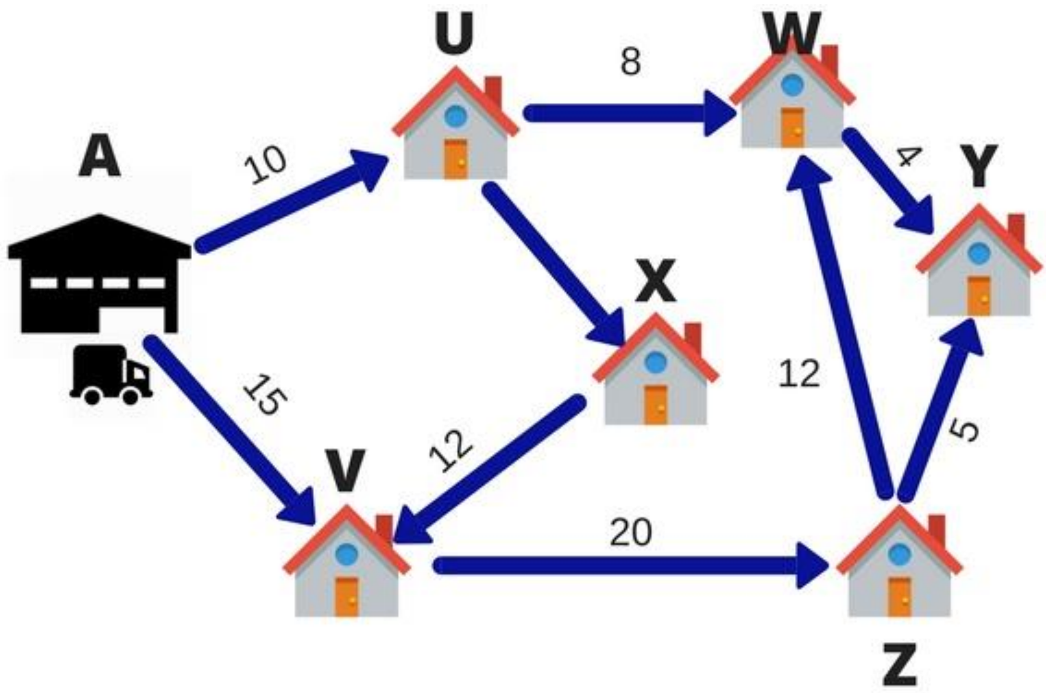
We first introduce three basic components essential for the development of LP theory.

Decision Variables: The variables in terms of which the problem is defined.

Objective Function: A function which is to be maximised or minimised subject to the given constraints/limitations.

Constraints: There are always certain limitations on the use of resources that limit the degree to which an objective can be achieved. These limitations are known as constraints or restrictions. Constraints must be represented as linear equalities or inequalities in terms of decision variables.

- Every organisation, big or small wants to find the best allocation of resources in order to optimize the objective function. We can use linear programming only if the following conditions are satisfied:
- Objective function should be well defined
- Objective function can be expressed as a linear function of the decision variables.
- There should be finite number of constraints and can be expressed as linear equalities or inequalities in terms of variables.
- Decision variables should be non- negative



1.2.1 Basic assumptions of LPP

Linearity: You need to express both the Objectives function and constraints as linear inequalities.

Deterministic: All co-efficient of decision variables in the Objectives and constraints expressions are known and finite.

Additive: The value of the Objectives function and the total sum of resources used must be equal to the sum of the contributions earned from each decision variable and the sum of resources used by decision variables respectively.

Divisibility: The solution of decision variables and resources can be non- negative values including fractions.

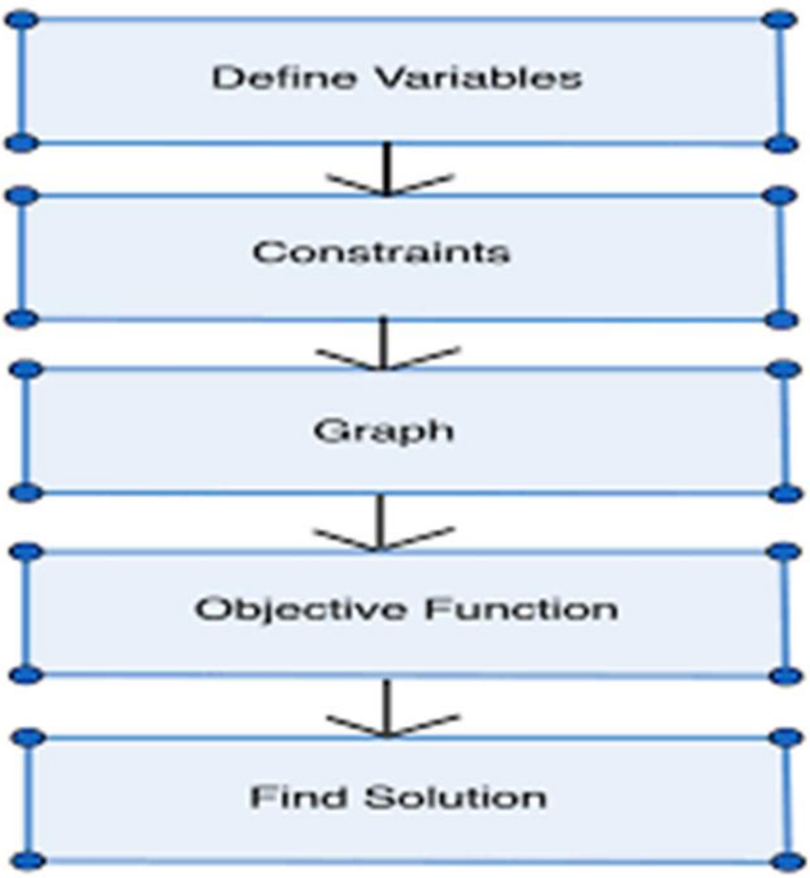
1.2.2 Formulation of LPP

Let us look at the steps of defining a Linear Programming problem generically:

1. Identify the decision variables
2. Write the objective function
3. Mention the constraints
4. Explicitly state the non-negativity restriction

For a problem to be a linear programming problem, the decision variables, objective function and constraints all have to be linear functions.

If all the three conditions are satisfied, it is called a Linear Programming Problem.



1.2.3 Types of Linear Programming Problems

Manufacturing Problems:

Manufacturing problems aim to optimize production decisions for maximum profits or minimal costs, considering resource availability (labor, materials), production rates, fees, and product selling prices.

Diet Problems:

Diet problems seek the least costly diet meeting nutritional requirements, factoring in nutritional content, food costs, and dietary constraints (allergies, preferences). A nutritionist may minimize costs while meeting nutritional needs.

Transportation Problems:

Transportation problems minimize moving goods costs from sources to destinations. Factors include supply at sources, demand at destinations, and transportation costs. Companies optimize shipping from factories to warehouses under constraints.

Optimal Assignment Problems:

Optimal assignment problems efficiently allocate tasks or resources. Factors include individual or machine skills and costs or time for different assignments. Managers may assign employees to projects for time optimization.

1.2.4 Steps in LPP formulation

- Identify decision variables
- Write objective function
- Formulate constraints

Formulation of Linear Programming Model

Step 1

From the study of the situation find the key-decision to be made. In the given situation key decision is to decide the extent of products 1, 2 and 3, as the extents are permitted to vary.

Step 2

Assume symbols for variable quantities noticed in step 1. Let the extents (amounts) of products 1, 2 and 3 manufactured daily be x1, x2 and x3 units respectively.

Step 3

Express the feasible alternatives mathematically in terms of variable. Feasible alternatives are those which are physically, economically and financially possible. In the given situation feasible alternatives are sets of values of x1, x2 and x3 units respectively.

where x1, x2 and x3 ≥ 0.

since negative production has no meaning and is not feasible.

Step 4

Mention the objective function quantitatively and express it as a linear function of variables. In the present situation, objective is to maximize the profit.

i.e., $Z = 4x_1 + 3x_2 + 6x_3$

Step 5

Put into words the influencing factors or constraints. These occur generally because of constraints on availability (resources) or requirements (demands). Express these constraints also as linear equations/inequalities in terms of variables.

Here, constraints are on the machine capacities and can be mathematically expressed as

$$2x_1 + 3x_2 + 2x_3 \le 440$$

$$4x_1 + 0x_2 + 3x_3 \le 470$$

1.2.5 Advantages and Limitations of an LPP

Linear programming methods are used in many fields including business and industry by almost all their departments such as production, marketing, finance etc. it's some advantages are

- Helps in attaining the optimum use of resources i.e. maximise profit and minimise costs
- Improve the quality of decisions

  There are many more advantages. In spite of having many advantages and wide areas of applications, there are some limitations as well. Following are certain limitations of linear programming:

We can apply linear programming method only if relationships are linear

- While solving LPP, it is possible that we will get non-integral values even for those decision variables which have only integral values.
- Constraints in the linear programming methods are written assuming all parameters are known and should be constant. However, in real problems, sometimes these are neither known nor constant.
- LP deals with the problems having single objective, whereas in real-life, there are many situations where we have to achieve multi-objectives.

Let us Sum Up

Rooted in lean principles derived from manufacturing, LPP aims to streamline processes and eliminate waste, thereby enhancing productivity and maximizing value delivery to customers. Central to LPP is the concept of iterative development cycles, where small, incremental changes are made based on rapid feedback loops.

1.2 Check Your Progress

1. What is a fundamental assumption of leaner programming?

- A) Maximizing complexity in software design
- B) Minimizing feedback from customers
- C) Continuous improvement through rapid iterations
- D) Increasing bureaucratic processes

2. Which statement best describes the formulation of leaner programming principles?

- A) Based on rigid, inflexible methodologies
- B) Rooted in lean manufacturing principles
- C) Emphasizing lengthy documentation processes
- D) Ignoring the need for iterative development

3. Leaner programming emphasizes:

- A) Lengthy planning phases
- B) Rapid feedback and adaptation

- C) Isolation from customer needs
- D) Complex and convoluted workflows

4. What is a core assumption about waste in leaner programming?

- A) Waste is unavoidable and should be ignored
- B) Waste can be identified and eliminated
- C) Waste should be maximized to ensure thoroughness
- D) Waste does not affect productivity

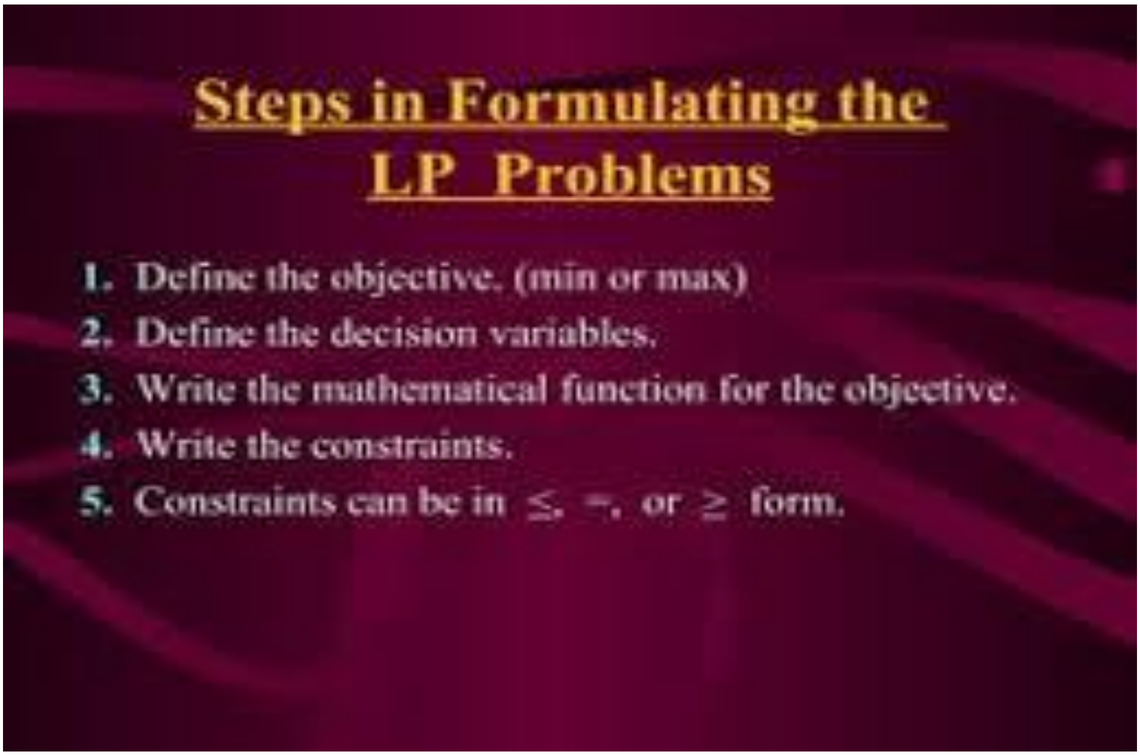
5. The formulation of leaner programming principles aims to:

- A) Increase complexity in development processes
- B) Decrease team collaboration
- C) Enhance efficiency and reduce waste
- D) Minimize customer engagement

1.3 Mathematical formulation of LPP

In our daily life, there are many real-life situations where LP problems may arise and for using LPP methods/techniques to find a solution of such situations, it becomes necessary to present the given word problem into mathematical form correctly. The steps of mathematical formulation of LPP are summarized as follows:

- Identify the decision variable of the given problem.
- Formulate the objective function, which is to be maximised or minimised, as a linear function of the decision variables.
- Formulate the constraints and express them as linear inequalities or equalities in terms of decision variables.
- Introduce non-negative restrictions as negative values of the decision variables do not have any valid physical interpretation.



1.3.1 Steps of LP model formulation

After formulating a linear programming problem, our next step is to solve it. You have learnt that linear programming problems can be represented as problems of maximisation or minimisation with constraints such as ≤ , =, ≥ . In order to develop a standard procedure for solving LPPs, we need to convert them into well - known form. We now discuss the General LPP along with these two forms. The canonical form is especially used in the duality theory and the standard form is used to develop the general procedure for solving any linear programming problem. In order to understand these forms, you also need to learn about slack and surplus variable.

1.3.2 General Linear Programming Problem-Model

Let us formulate the general linear programming problem. Let Z be a linear function of n basic variables

$X1,2, X3,..., Xn$, which is to be maximised (or minimised). We write the problem as Maximise (or minimise) Z = $C1X1 + C2X2 + C3X3 + ... + CnXn$

.... (1)

where $C1$, $C2$, $C3$, , Cn are known constant termed as cost coefficients of basic variables.

Let (aij) be an m×n real matrix of m×n constants aij's and let {$b1$,, $b2$, … , bm} be a set of constants such that

$a11x1$ + $a12x2$ + ... + $a1nxn$ ≤ or = or , ≥ $b1$

$a21x1$ + $a22x2$ + ... + $a2nxn$ ≤ or = or , ≥ $b2$

. . . . (2)

. . . .

. . . .

$am1x1$ + $am2x2$ + ... + $amnxn$ ≤ or = or, ≥ bm

xj≥ 0 for all j = 1 ,2, 3, ..., n ... (3)

And

The linear function Z in equation (1) is called the objective function. The set of inequalities given in (2) is called constraints of a general LPP and the set of inequalities given in (3) are known as non – negative restrictions of a general LPP. Slack and Surplus Variables

In general, if any linear programming problem, we have a constraint of the type

$a11x1$ + $a12x2$ + ... + $a1nxn$ ≤ $b1$ where $b1$≥ 0

Then this inequality can be converted into an equation by adding one non – negative variable $s1$ to the left-hand side. This new variable is called a slack variable and the constraints are transformed into the following equation:

$a11x1$ + $a12x2$ + ... + $a1nxn$ + $s1$ = $b1$ where $s1$≥ 0, $b1$≥ 0

Thus, a non – negative variable subtracted from the left – hand side of less than or equal to (≤) type of a constraint that converts it into an equation is called a slack variable. The values of this variable can be interpreted as the amount of unused resource.

Similarly, if in any linear programming problem, we have a constraint of the type

$a11x1$ + $a12x2$ + ... + $a1nxn$≥ $b1$

Then this inequality can be converted into an equation by subtracting one non –negative variable $s1$ from the left-hand side. This new variable is called a surplus variable. The value if this variable can be interpreted as the amount over and above the required minimum level.

1.4 CANONICAL FORM

A linear program in its canonical form is:

- A Maximization problem, under Lower or equal constraints, all the variables of which are strictly positive.
- A problem of Minimization, under Greater or equal constraints, all of whose variables are strictly positive.

If the linear program does not correspond to these criteria, it is necessary to transform the constraints or the objective function according to the following operations:

- max z = - min -z
- x + y ≥ b is equivalent to - x - y ≤ - b
- x + y = b is equivalent to x + y ≥ b, x + y ≤ b

The canonical form is often represented in a matrix form:

- the vector of the coefficients of the objective function: c of size n
- the matrix of the coefficients of the left part of the constraints: A of size m * n
- the vector of the constants of the right part of the constraints: b of size m
- the vector of variables: x of size n

Thus the following linear program:

$$\begin{cases} max & c_1x_1 & + & \dots & + & c_ix_i & + & \dots & + & c_nx_n & & \\ s.c. & a_{11}x_1 & + & \dots & + & a_{1i}x_i & + & \dots & + & a_{1n}x_n & \leq & b_1 \\ & \vdots & & & & \vdots & & & & \vdots & & \vdots \\ & a_{j1}x_1 & + & \dots & + & a_{ji}x_i & + & \dots & + & a_{jn}x_n & \leq & b_j \\ & \vdots & & & & \vdots & & & & \vdots & & \vdots \\ & a_{m1}x_1 & + & \dots & + & a_{mi}x_i & + & \dots & + & a_{mn}x_n & \leq & b_m \end{cases}$$

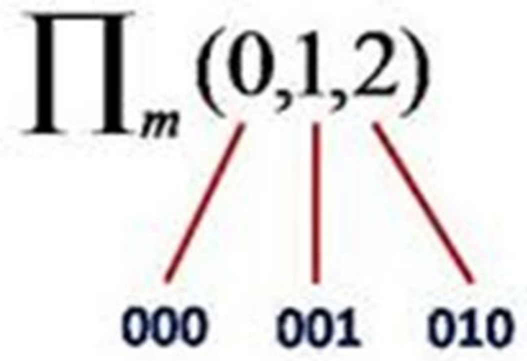
Written in the following form:

$$\max_{\mathbf{x}} \left[F(\mathbf{x}) = \mathbf{c}^{\top}\mathbf{x} = c_1 x_1 + \cdots c_n x_n \right]$$

sous les contraintes :

$$\begin{cases} A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases}$$

POS form



$$(A + B + C)(A + B + \overline{C})(A + \overline{B} + C)$$

1.4.1 The characteristics of the canonical form are:

- Objective function should be of maximization form. If it is given in minimization form, it should be converted into maximization form.
- All the constraints should be of "≤ " type, except for non- negative restrictions. Inequality of " ≥ " type, if any, should be changed to an inequality of the " ≤ " type.

All variables should be non-negative. If a given variable is unrestricted in sign (i.e., positive, negative or zero), it can be written as a difference of two non-negative variables. Suppose x is unrestricted in sign, then x can be written as x= x'- x'' where $x' \geq 0$, $x'' \geq 0$

1.6 Graphical method of solution TO LPP

A linear program can be solved by multiple methods. In this section, we are going to look at the Graphical method for solving a linear program. This method is used to solve a two-variable linear program. If you have only two decision variables, you should use the graphical method to find the optimal solution.

A graphical method involves formulating a set of linear inequalities subject to the constraints. Then the inequalities are plotted on an X-Y plane. Once we have plotted all the inequalities on a graph the intersecting region gives us a feasible region. The feasible region explains what all values our model can take. And it also gives us the best solution.

Let's understand this with the help of an example.

Illustration 1

A farmer has recently acquired a 110 hectares piece of land. He has decided to grow Wheat and barley on that land. Due to the quality of the sun and the region's excellent climate, the entire production of Wheat and Barley can be sold. He wants to know how to plant each variety in the 110 hectares, given the costs, net profits and labor requirements according to the data shown below:

Variety	Cost (Price/Hec)	Net Profit (Price/Hec)	Man-days/Hec
Wheat	100	50	10
Barley	200	120	30

The farmer has a budget of US$10,000 and availability of 1,200 man-days during the planning horizon. Find the optimal solution and the optimal value.

Formulation of a Linear Problem

Step 1: Identify the decision variables

The total area for growing Wheat = X (in hectares)

The total area for growing Barley = Y (in hectares)

X and Y are my decision variables.

Step 2: Write the objective function

Since the production from the entire land can be sold in the market. The farmer would want to maximize the profit for his total produce. We are given net profit for both Wheat and Barley. The farmer earns a net profit of US$50 for each hectare of Wheat and US$120 for each Barley.

Our objective function (given by Z) is, **Max Z = 50X + 120Y**

Step 3: Writing the constraints

1. It is given that the farmer has a total budget of US$10,000. The cost of producing Wheat and Barley per hectare is also given to us. We have an upper cap on the total cost spent by the farmer. So our equation becomes:

100X + 200Y ≤ 10,000

2. The next constraint is the upper cap on the availability of the total number of man-days for the planning horizon. The total number of man-days available is 1200. As per the table, we are given the man-days per hectare for Wheat and Barley.

10X + 30Y ≤ 1200

3. The third constraint is the total area present for plantation. The total available area is 110 hectares. So the equation becomes,

X + Y ≤ 110

Step 4: The non-negativity restriction

The values of X and Y will be greater than or equal to 0. This goes without saying.

X ≥ 0, Y ≥ 0

We have formulated our linear program. It's time to solve it.

Solving an LP Through the Graphical Method

Since we know that X, Y ≥ 0. We will consider only the first quadrant.

To plot for the graph for the above equations, first I will simplify all the equations.

100X + 200Y ≤ 10,000 can be simplified to X + 2Y ≤ 100 by dividing by 100.

10X + 30Y ≤ 1200 can be simplified to X + 3Y ≤ 120 by dividing by 10.

The third equation is in its simplified form, X + Y ≤ 110.

Plot the first 2 lines on a graph in the first quadrant (like shown below)

The optimal feasible solution is achieved at the point of intersection where the budget & man-days constraints are active. This means the point at which the equations X + 2Y ≤ 100 and X + 3Y ≤ 120 intersect gives us the optimal solution.

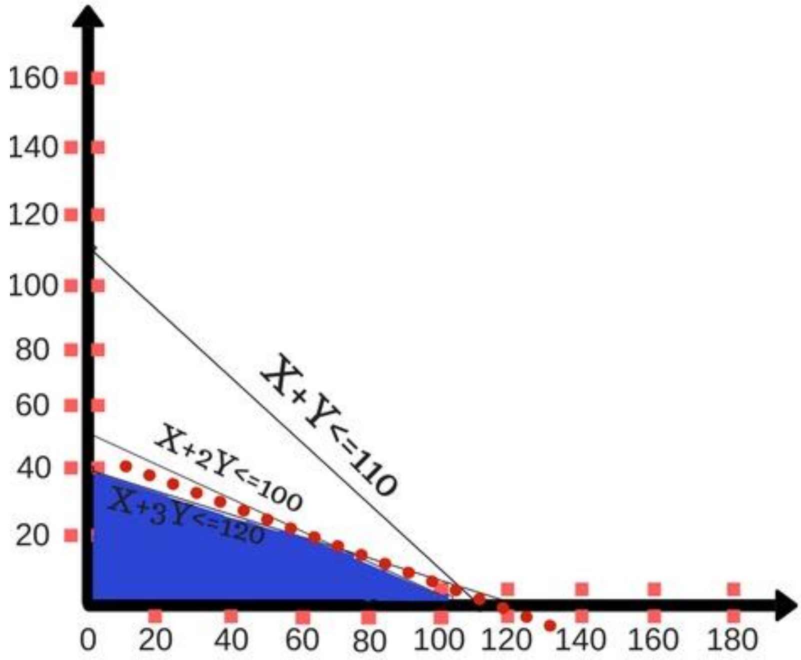
The values for X and Y which gives the optimal solution is at (60,20).

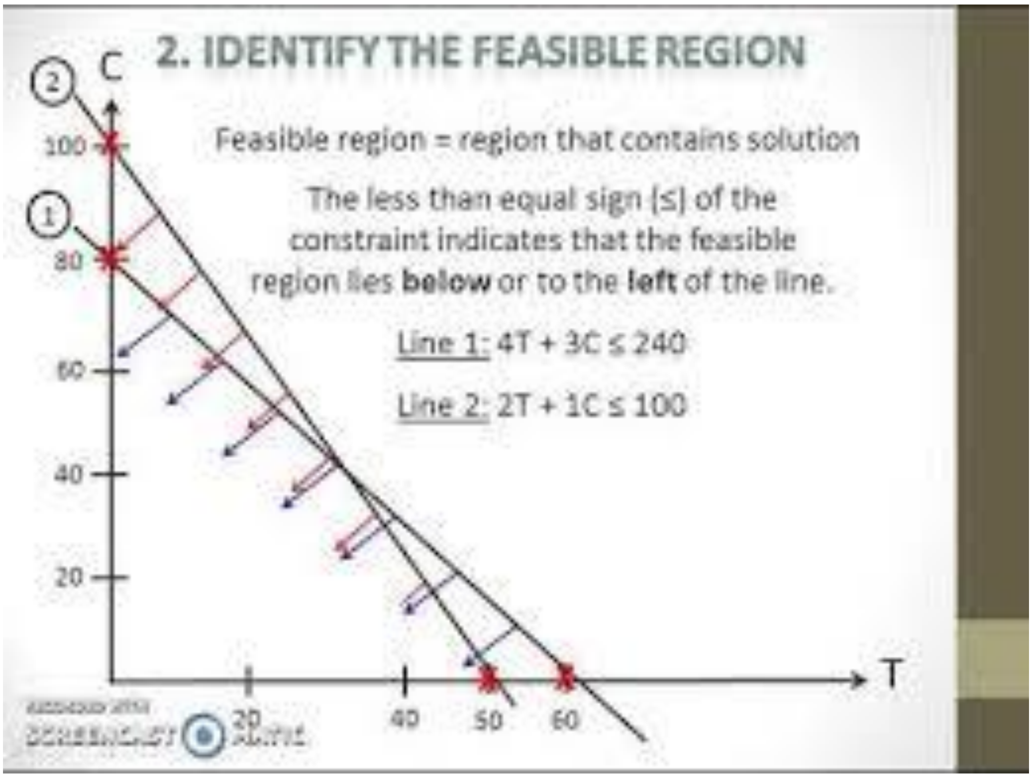
To maximize profit the farmer should produce Wheat and Barley in 60 hectares and 20 hectares of land respectively.

The maximum profit the company will gain is,

Max Z = 50 * (60) + 120 * (20)

= US$5400





In this we use two models

1. Maximization Problems
2. Minimization problems

1.6.1 Maximization Problem

A typical **linear programming** problem consists of finding an extreme value of a linear function subject to certain constraints. We are either trying to maximize or minimize the value of this linear function, such as to maximize profit or revenue, or to minimize cost. That is why these linear programming problems are classified as **maximization** or **minimization problems**, or just **optimization problems**. The function we are trying to optimize is called an **objective function**, and the conditions that must be satisfied are called **constraints**.

Illustration

Maximize Z = 30X1 + 40X2

Subject to 3X1 + 2X2 ≤ 600

$3X1 + 5X2 \leq 800$

$5X1 + 6X2 \leq 1100$ and

$X1 \geq 0, X2 \geq 0$

Solution:-

Let us consider Equation (1) i.e

Put X1 = 0, X2 = 0, Equation (1)

$Z = 30X1 + 40X2$

$3X1 + 2X2 \leq 600$

We can get point = (0, 0)

Put X1 = 0, in Equation (1) X2= 300

and X2 = 0, in Equation (1) X1= 200

We can get points = (0, 300) and (200,0)

Let us consider Equation (2) i.e

$3X1 + 5X2 \leq 800$

Put X1 = 0, in Equation (2) X2= 160 and X2 = 0, in Equation (2) X1= 266.66

We can get points = (0, 160) and (266.66, 0)

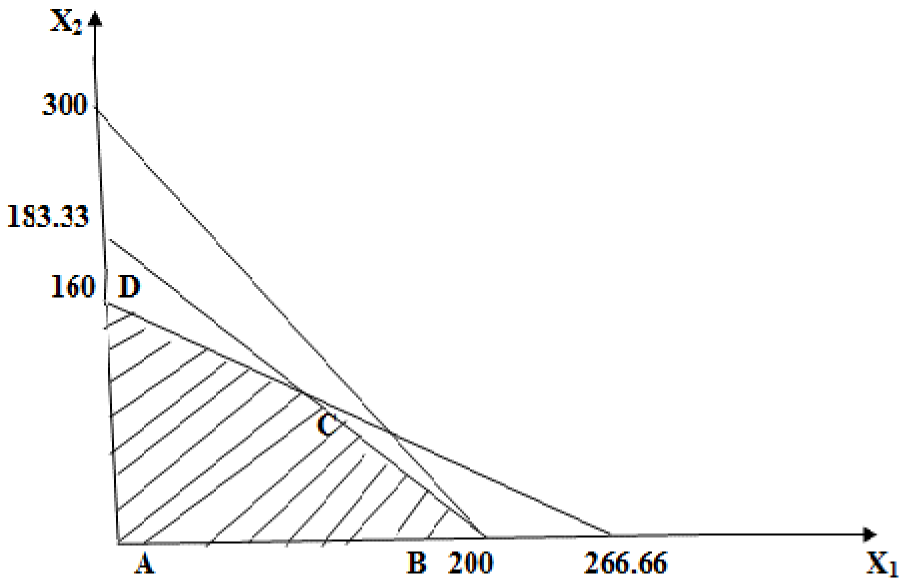
Let us consider Equation (3) i.e

$5X1 + 6X2 \leq 1100$

Put X1 = 0, in Equation (3) X2= 183.33 and X2 = 0, in Equation (3) X1=220

We can get points = (0, 183.33) and (200, 0)

A, B, C, and D is feasible Region



With help of graph from the Feasible Region we calculate Z MAX Value

Key Terms:

Objective Function: Is a linear function of the decision variables representing the objective of the manager/decision maker.

Constraints: Are the linear equations or inequalities arising out of practical limitations.

Decision Variables: Are some physical quantities whose values indicate the solution.

Feasible Solution: Is a solution which satisfies all the constraints (including the non-negative) presents in the problem.

Feasible Region: Is the collection of feasible solutions.

Multiple Solutions: Are solutions each of which maximize or minimize the objective function.

Unbounded Solution: Is a solution whose objective function is infinite.

Infeasible Solution: Means no feasible solution.

1.6.2 Maximization Problem

A typical example is to maximize profit from producing several products, subject to limitations on materials or resources needed for producing these items; the problem requires us to determine the amount of each item produced. Another type of problem involves scheduling; we need to determine how much time to devote to each of several activities in order to maximize income from (or minimize cost of) these activities, subject to limitations on time and other resources available for each activity.

Illustration:

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation.

If Nikki makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

We start by choosing our variables.

- Let xx = The number of hours per week Niki will work at Job I.
- Let yy = The number of hours per week Niki will work at Job II.

Now we write the objective function. Since Niki gets paid $40 an hour at Job I, and $30 an hour at Job II, her total income I is given by the following equation.

I=40x+30y$I=40x+30y$

Our next task is to find the constraints. The second sentence in the problem states, "She never wants to work more than a total of 12 hours a week." This translates into the following constraint:

x+y≤12$x+y\leq 12$

The third sentence states, "For every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation." The translation follows.

2x+y≤16$2x+y\leq 16$

The fact that xx and yy can never be negative is represented by the following two constraints:

x≥0, and y≥0.$x\geq 0$, and $y\geq 0$.

Well, good news! We have formulated the problem. We restate it as

Maximize Subject to: I=40x+30yx+y≤122x+y≤16x≥0;y≥0(3.1.1)(3.1.1) Maximize I=40x+30y Subject to: x+y≤122x+y≤16x≥0;y≥0

In order to solve the problem, we graph the constraints and shade the region that satisfies **all** the inequality constraints.

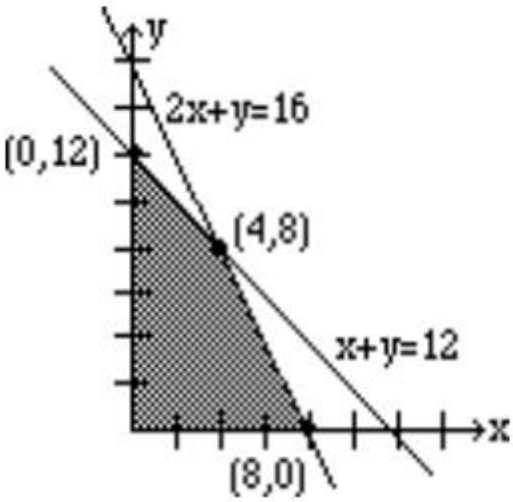
Any appropriate method can be used to graph the lines for the constraints. However often the easiest method is to graph the line by plotting the x-intercept and y-intercept.

The line for a constraint will divide the plane into two region, one of which satisfies the inequality part of the constraint. A test point is used to determine which portion of the plane to shade to satisfy the inequality. Any point on the plane that is not on the line can be used as a test point.

- If the test point satisfies the inequality, then the region of the plane that satisfies the inequality is the region that contains the test point.
- If the test point does not satisfy the inequality, then the region that satisfies the inequality lies on the opposite side of the line from the test point.

In the graph below, after the lines representing the constraints were graphed using an appropriate method from Chapter 1, the point (0,0) was used as a test point to determine that

- (0,0) satisfies the constraint x+y≤12$x+y$≤12 because 0+0<120+0<12
- (0,0) satisfies the constraint 2x+y≤162$x+y$≤16 because 2(0)+0<16
- Therefore, in this example, we shade the region that is below and to the left of both constraint lines, but also above the x axis and to the right of the y axis, in order to further satisfy the constraints x≥0x≥0 and y≥0y≥0.



he shaded region where all conditions are satisfied is called the **feasibility region** or the feasibility polygon.

The **Fundamental Theorem of Linear Programming** states that the maximum (or minimum) value of the objective function always takes place at the vertices of the feasibility region.

Therefore, we will identify all the vertices (corner points) of the feasibility region. We call these points **critical points.** They are listed as (0, 0), (0, 12), (4, 8), (8, 0). To maximize Niki's income, we will substitute these points in the objective function to see which point gives us the highest income per week. We list the results below.

Critical Points	Income
(0, 0)	40(0) + 30(0) = $0
(0, 12)	40(0) + 30(12) = $360
(4, 8)	40(4) + 30(8) = $400
(8, 0)	40(8) + 30(0) = $320

Clearly, the point (4, 8) gives the most profit: $400.

Therefore, we conclude that Niki should work 4 hours at Job I, and 8 hours at Job II.

Let us Sum Up

Canonical form and graphical form are two key representations used in linear programming (LP) to model and solve optimization problems. On the other hand, the graphical form of LP provides a visual representation of the problem's constraints and objective function in a two-dimensional or occasionally three-dimensional space. In graphical form, variables are typically represented on axes, and constraints are plotted as lines or planes defining boundaries within which feasible solutions exist.

1.3 Check your Progress

1. In the canonical form of linear programming, the objective is to:

A) Maximize a linear function
B) Minimize a quadratic function
C) Minimize a linear function
D) Maximize a non-linear function

2. The graphical model in linear programming represents:

A) Non-linear constraints
B) Feasible regions and objective function
C) Constraints as inequalities only
D) Multiple objective functions

3. What is the primary goal of minimization in linear programming?

A) Maximizing waste
B) Minimizing profit
C) Minimizing costs or resource utilization
D) Minimizing team efficiency

4. Which form of LP representation provides a visual depiction of constraints and the objective function?

A) Canonical form
B) Algebraic form
C) Graphical form
D) Matrix form

5. The canonical model in linear programming is characterized by:

A) Graphical representations only
B) Quadratic constraints
C) Linear constraints and objectives
D) Non-linear objectives

GLOSSARY

- Decision Variables: The decision variables are the variables that will decide my output. They represent my ultimate solution. To solve any problem, we first need to identify the decision variables. For the above example, the total number of units for A and B denoted by X & Y respectively are my decision variables.
- Objective Function: It is defined as the objective of making decisions. In the above example, the company wishes to increase the total profit represented by Z. So, profit is my objective function.
- Constraints: The constraints are the restrictions or limitations on the decision variables. They usually limit the value of the decision variables. In the above example, the limit on the availability of resources Milk and Choco are my constraints.
- Non-negativity Restriction: For all linear programs, the decision variables should always take non-negative values. This means the values for decision variables should be greater than or equal to 0.

Self-assessment Questions

1.Let us consider a company making single product. The estimated demand for the product for the next four months are 1000,800,1200,900 respectively. The company has a regular time capacity of 800 permonth and an overtime capacity of 200 per month. The cost of regular time production is Rs.20 per unit and the cost of overtime production is Rs.25 per unit. The company can carry inventory to the next month and the holding cost is Rs.3/unit/month the demand has to be met every month. Formulate a linear programming problem for the above situation.

2. What are the advantages and applications of OR ?

3. Solve the following LPP by Big-M penalty method Minimize Z = 5 X1 + 3 X2

S.T 2 X1 + 4 X 2 =1 2, 2 X1 + 2 X 2 = 1 0 ,5 X1 +2 X 2 = 1 0 and X1 , X 2 ≤ 0

4. Solve the following LP problem using graphical method

Maximize Z == -X1+2X2

Subjected to X1-X2≤-1

-0.5X1 - X2≤2 x1,x2≥ 0.

Exercise

5. Three foods with the following nutrient contents and costs:

Food 1: 2 units of protein, 1 unit of fat, 3 units of carbohydrates, cost: $5

Food 2: 1 unit of protein, 2 units of fat, 1 unit of carbohydrates, cost: $3

Food 3: 3 units of protein, 1 unit of fat, 2 units of carbohydrates, cost: $8

Requirements: At least 5 units of protein, 4 units of fat, and 6 units of carbohydrates. Minimize the cost of the diet while meeting the nutritional requirements.

6. The company needs different numbers of workers on different days:

Monday: 5

Tuesday: 3

Wednesday: 4

Thursday: 4

Friday: 6

Saturday: 2

Sunday: 2

Each worker works 5 consecutive days and has 2 days off. Minimize the number of workers required while meeting daily needs.

7. Two products with the following production times and profits:

Product A: 2 hours per unit, profit: $30

Product B: 3 hours per unit, profit: $40

Constraints: 120 production hours available. Maximize the total profit.

8. Three factories and three warehouses with the following supply and demand:

Factories: A (20 units), B (30 units), C (50 units)

Warehouses: X (40 units), Y (35 units), Z (25 units)

Transportation costs per unit:

A to X: $2, A to Y: $3, A to Z: $1

B to X: $4, B to Y: $2, B to Z: $5

C to X: $3, C to Y: $4, C to Z: $2, Minimize the total transportation cost.

9. Three types of raw materials to produce two products. Each raw material has a different cost and contribution to the products:

- Raw Material 1: $10/unit
- Raw Material 2: $15/unit
- Raw Material 3: $12/unit
- Product A requires: 2 units of Raw 1, 1 unit of Raw 2
- Product B requires: 1 unit of Raw 1, 2 units of Raw 3
- Product demands: 50 units of A, 30 units of B, Minimize the cost of raw materials.

10. Four investment options with different expected returns and risks:

- Investment 1: 8% return, 5% risk
- Investment 2: 10% return, 7% risk
- Investment 3: 12% return, 6% risk
- Investment 4: 14% return, 8% risk
- Constraints: Total investment $100,000, maximum average risk 6%., Maximize the total return.

Answer for Check your Progress

1.1

1. C) Enhancing team productivity and efficiency
2. C) Continuous improvement and waste reduction
3. C) Increasing bureaucracy
4. B) Rapid iteration and feedback
5. B) By reducing costs and increasing competitiveness

1.2

1. C) Continuous improvement through rapid iterations
2. B) Rooted in lean manufacturing principles
3. B) Rapid feedback and adaptation
4. B) Waste can be identified and eliminated
5. C) Enhance efficiency and reduce waste

1.3

1. A) Maximize a linear function
2. B) Feasible regions and objective function
3. C) Minimizing costs or resource utilization
4. C) Graphical form
5. C) Linear constraints and objectives

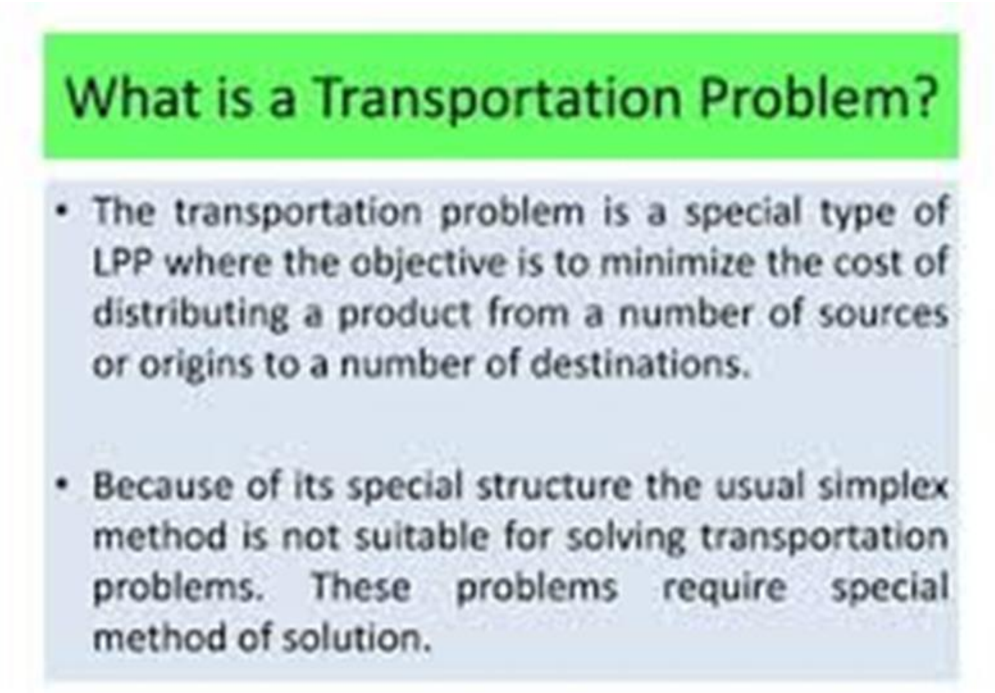
UNIT II

Unit Objective

The objective of the transportation problem in operations research is to determine the most efficient way to distribute a product from multiple suppliers to multiple consumers while minimizing the total transportation cost. This involves finding the optimal allocation of shipments from each supplier to each consumer that satisfies the demand at each consumer location and the supply at each supplier location. The transportation problem seeks to minimize the total cost of shipping goods, which is calculated based on the unit transportation costs associated with each route, the supply quantities available at each origin, and the demand quantities required at each destination. By solving this problem, businesses can achieve cost-effective distribution strategies, improving their overall operational efficiency and resource utilization

2.1 TRANSPORTATION PROBLEM

Transportation problem is a special kind of Linear Programming Problem (LPP) in which the objective is to transport goods from a set of sources/origins to a set of destinations in such a manner that the total transportation or shipping cost is minimized. To achieve this objective, we must know about some parameters such as the quantity of available supplies, the quantity demanded and the costs of shipping a unit from various origins to various destinations.



2.1.1Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

Step 1: Determine a starting basic feasible solution, using any one of the following three Methods

1. North West Corner Method
2. Least Cost Method
3. Vogel Approximation Method

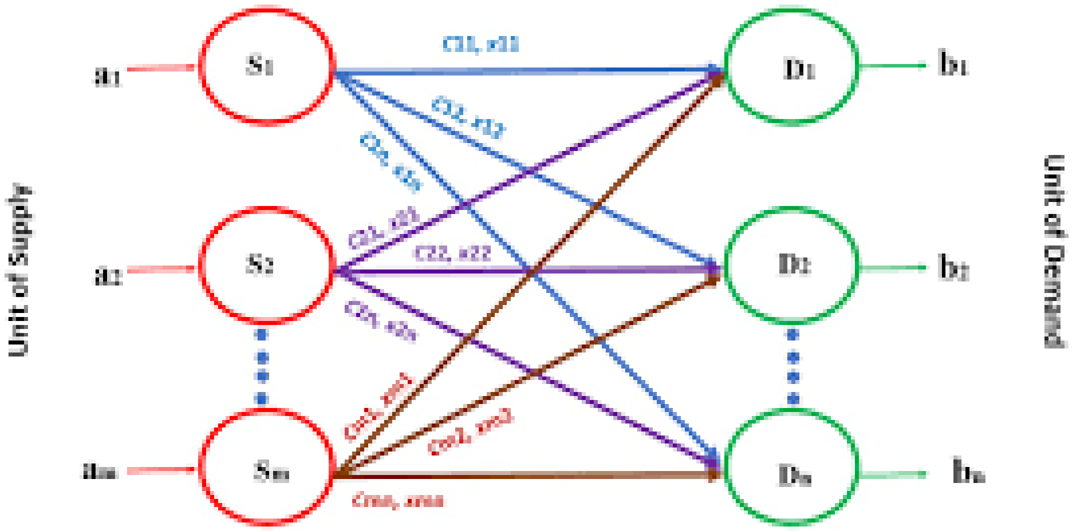
Step 2: Determine the optimal solution using the following method

1. MODI (Modified Distribution Method) or UV Method.

2.1.2 Mathematical formulation of the transportation problem

Let there be m origins/ sources of supply $O_1, O_2, \ldots, O_i \ldots O_m$ and n destinations$D_1, D_2, \ldots, D_j \ldots D_n$. The total number of the capacities of all m origins is assumed to be equal to the total number of the requirements of all n destinations. Let C_{ij} be the cost of shipping one unit from origin i to destination j. Let a_i be the capacity/ availability of items at origin i and b_j, the requirement/demand of the destination

j. Then this transportation problem can be expressed in a tabular form as follows:



TRANSPORTATION PROBLEM

Origin	Destinations	Availability/ capacity
	D_1 D_1 … D_j … D_n	

O_1	C_{11}	C_{12}	…	C_{1j}	…	C_{1n}	a_1
	C_{21}	C_{22}	…	C_{2j}	…	C_{2j}	
O_2	.	.	.	.			a_2
	.	.	.	.			.
	.	.	.	.			.
O_i	C_{i1}	C_{i2}	…	C_{ij}	…	C_{in}	.
	.	.	.	.			a_i
	.	.	.	.			.
O_m	.	.	.	.			.
	C_{m1}	C_{m2}	…	C_{mj}	…	C_{mn}	.
							a_m
Requirement/ Demand	b_1	b_2	…	b_j	…	b_n	Total

The condition for the existence of a feasible solution to a transportation problem is give as

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The above equation tells us that the total requirement/demand equals the total capacity. If it is not so, a dummy origin or destination is created to balance the total capacity and requirement.

Now let x_{ij} be the number of units to be transported from origin i to destination j and C_{ij} the corresponding cost of transportation. Then the total transportation cost is $\sum_{i=1}^{m} . \sum_{j=1}^{n} C_{ij}x_{ij}$

Subject to the constraints:

$$\sum_{j=1}^{n} x_{1j} = a_1, \sum_{j=1}^{n} x_{2j} = a_2, \ldots, \sum_{j=1}^{n} x_{mj} = a_m \qquad \ldots\ldots(3)$$

$$\sum_{i=1}^{m} x_{i1} = b_1, \sum_{i=1}^{m} x_{i2} = b_2, \ldots, \sum_{j=1}^{n} x_{in} = b_n$$

And $x_{ij} \geq 0$ for all i = 1, 2, 3... m and j = 1, 2... n.

The Simplex method is regarded as the most generalized method to solve this. However, the solution is very lengthy and takes a long time to solve it since a large number of decision variables and artificial variables are involved. It is far simpler to solve it by transportation method as compared to the Simplex method. In the transportation method, we first obtain the initial basic feasible solution and then perform the optimality test.

$j =$

2.1.3 Methods of finding initial basic feasible solution

There are several methods to obtain initial basic feasible solution. Here, we shall discuss the following methods to determine the initial basic feasible solution:

(i) North-West Corner Rule

(ii) Least Cost Method

(iii) Vogel's Approximation Method (Penalty or Regret Method)

Vogel's Approximation method generally gives a solution closer to the optimum solution. Hence, it is preferred to the other two methods.

Let Us Sum Up

Introduction to transportation problems and its mathematical calculation were discussed in earlier chapters.

1.1 Check Your Progress

1. Which of the following methods is commonly used to find an initial feasible solution in a transportation problem?
   - A) Northwest Corner Rule
   - B) Simplex Method
   - C) Hungarian Method
   - D) Johnson's Rule

2. In the context of the transportation problem, what does 'degeneracy' refer to?
   - A) A situation where supply equals demand
   - B) A situation where there are more basic variables than required
   - C) A situation where the total cost is minimized
   - D) A situation where some of the transportation routes are not used

3. What is the main objective of the transportation problem?
   - A) Maximizing the supply
   - B) Minimizing the transportation time
   - C) Minimizing the total transportation cost
   - D) Balancing the supply and demand

4. Which of the following conditions must be satisfied for a transportation problem to be balanced?
   - A) Total supply is greater than total demand
   - B) Total supply is less than total demand
   - C) Total supply equals total demand
   - D) There is no relationship between supply and demand

5. Which method is used to check the optimality of a transportation problem's solution?
   - A) Vogel's Approximation Method (VAM)
   - B) MODI (Modified Distribution) Method
   - C) Least Cost Method
   - D) Critical Path Method (CPM)

2.2 North – West Corner Rule

The North – West Corner Rule (NWC)is a simple and efficient method to obtain initial basic feasible solution. North-west corner rule is one of the easiest methods to find a feasible solution to a transportation problem. Before getting into detail about the North-west corner rule, let's recall what a transportation problem is.

Transportation problem:

It is a special type of Linear Programming Problem (LPP) in which goods are transported from one set of sources to another set of destinations based on the supply and demand of the origins and destination, respectively, such that the total cost of transportation is minimized.

There are two types of transportation problems. They are

Balanced transportation problems: Both supply and demand are equal.

Unbalanced transportation problem: Supply and demand are not equal. In this case, either a dummy column or row is added based on the necessity to make it a balanced problem.

It can be summarized as follows:

Step 1: Start with cell (1, 1) at the north-west corner (upper left-hand corner) of the transportation matrix and allocate as much as possible there.

Step 2: Here, we have three cases

(a) If the quantity needed at First Destination (b_1) is less than the quantity available at First Origin (a_i), we allocate a quantity equal to the requirement at First Destination to the cell (1, 1). At

this stage, Column 1 is exhausted, so we cross it out. Since the requirement b_1 is fulfilled, we reduce the availability a_1 by b_1 and proceed to north-west corner of the resulting matrix, i.e., cell (1,2).

(b) If the quantity needed at First Destination (b_1) is greater than the quantity available at First Origin(a_1), allocate a quantity equal to the quantity available at First Plant/Origin (a_1) $to\ cell$ (1, 1).At this stage, Row 1 is exhausted, so we cross it out and proceed to north- west corner of the resulting matrix, i.e., cell (2, 1).

(c) If the quantity needed at First Destination (b_1) is equal to the quantity available at First Origin (a_1), we allocate a quantity equal to the requirement at First Destination (or the quantity available at First Origin). At this stage, both column 1 as well as Row 1 is exhausted. We cross them out and proceed to the north-west corner of the resulting matrix, i.e., cell (2, 2).

Step 3: We continue the procedure, until we reach the south – east corner of the original matrix.

Illustration 1. Find the basic feasible solution of the given transportation problem by applying North – West Corner rule:

Warehouse / Factory	D	E	F	G	Capacity
A	42	48	38	37	160
B	40	49	52	51	150
C	39	38	40	43	190
Requirement	80	90	110	220	500

Solution. We start from the North – West corner, i.e., the Factory A and Warehouse D. The quantity needed at the First Warehouse (Warehouse D) is 80, which is less than the quantity available (160) at the First Factory A. Therefore, a quantity equal to the warehouse D is to be allocated to the cell (A, D). Thus, the requirement of Warehouse D is met by Factory A. So, we cross out column 1 and reduce the capacity of Factory A by 80. Then we go to cell (A, E), which is North – West corner of the resulting matrix.

Now, the quantity needed at the second Warehouse (Warehouse E) is 90, which is greater than the quantity available (80) at the First Factory A. Therefore, we allocate a quantity equal to the capacity at Factory A, i.e., 80 to the cell (A, E). The requirement of Warehouse E is reduced to 10. The capacity of Factory A is exhausted and has to be removed from the matrix. Therefore, we cross out row 1 and proceed to cell (B, E).

Now, the quantity needed at the second Warehouse (Warehouse E) is 10, which is less than the quantity available at the Second Factory B, which is 150. Therefore, the quantity 10 equal to the requirement at Warehouse E is allocated to the cell (B, E). Hence, the requirement of Warehouse E is met and we cross out column 1. We reduce the capacity of Factory B by 10 and proceed to cell (B, F).

Again, the quantity needed at the Third Warehouse (Warehouse F) is 110. It is less than the quantity available at the Second Factory (Factory B), which is140. Therefore, a quantity equal to the requirement at Warehouse F is allocated to the cell (B, F). Since the requirement of Warehouse F is met, we cross out Column 1 and reduce the capacity of Factory B by 110. Then we proceed to cell (B, G).Now, the quantity needed at the Fourth Warehouse (Warehouse G) is 220, which is greater than the quantity available at the Second Factory (Factory B).Therefore, we allocate the quantity equal to the capacity of Factory B to the cell (B, G) so that the capacity of Factory B is exhausted and the requirement of Warehouse G is reduced to 190. Hence, we cross out Row 1 and proceed to cell (C, G).

Thus, the allocations given using North – West corner rule are as shown in the following matrix along with the cost per unit of transportation:

Warehouse / Factory	D	E	F	G	Capacity
A	42 [80]	48 [80]	38	37	160
B	40	49 [10]	52 [11]	51 [30]	150
C	39	38	40	43 [190]	190
Requirement	80	90	110	220	500

Thus, the total transportation cost for these allocations

= 42 × 80 + 48 × 80 + 49 × 10 + 52 × 110 + 51 × 30 + 43 × 190

= 3360 + 3840 + 490 + 5720 + 1530 + 8170 =23110

Illustration 2: Find the basic feasible solution of the following problem using North-West Corner Rule:

Origin/ Distribution Centre	1	2	3	4	5	6	Availability
1	4	6	9	2	7	8	10
2	3	5	4	8	10	0	12
3	2	6	9	8	4	13	4
4	4	4	5	9	3	6	18
5	9	8	7	3	2	14	20
Requirements	8	8	16	3	8	21	

Answer. Using North - West corner rule, the allocations are to be made as under:

units to cells (1,1), 2 units to cell (1,2), 6 units to cell (2,2), 6 units to cell (2,3), 4 units to cell (3 ,3), 6 units to cell (4,3), 3 units to cell (4,4), 8 units to cell (4,5), 1 unit to cell (4,6) and 20 units to cell (5,6) and the transportation cost is equals to 501

2.2.1 North West Corner Rule Steps

NORTH - WEST CORNER RULE

Step1:Identify the cell at North-West corner of the transportation matrix.

Step2:Allocate as many units as possible to that cell without exceeding supply or demand; then cross out the row or column that is exhausted by this assignment

Step3:Reduce the amount of corresponding supply or demand which is more by allocated amount.

Step4:Again identify the North-West corner cell of reduced transportation matrix.

Step5:Repeat Step2 and Step3 until all the rim requirements are satisfied.

Go through the steps given below to understand how to find a feasible solution for a transportation problem.

Step 1: Select the upper-left cell, i.e., the north-west corner cell of the transportation matrix and assign the minimum value of supply or demand, i.e., min(supply, demand).

Step 2: Subtract the above minimum value from Oi and Di of the corresponding row and column. Here, we may get three possibilities, as given below. If the supply is equal to 0, strike that row and move down to the next cell. If the demand equals 0, strike that column and move right to the next cell. If supply and demand are 0, then strike both row and column and move diagonally to the next cell.

Step 3: Repeat these steps until all the supply and demand values are 0.

North West Corner Method Solved Example

Illustration 1

Get an initial basic feasible solution to the given transportation problem using the North-west corner rule.

Solution:

For the given transportation problem, total supply = 950 and total demand = 950.

Thus, the given problem is the balanced transportation problem.

Now, we can proceed with the North-west corner rule to find the initial feasible solution.

Step 1: Consider the upper-left corner cell, which has the value 11. The minimum value of the corresponding cell's supply and demand is 200.

Step 2: The difference between the corresponding cell's supply and demand from the minimum value obtained in the previous step is:

Supply = 250 – 200 = 50

Demand = 200 – 200 = 0

As demand is 0, we need to allocate 200 to that cell and strike the corresponding column and then move right to the next cell, i.e., the cell with the value 13.

Step 3: For the cell with value 13, the minimum of supply and demand is min(50, 225) = 50.

Step 4: The difference between the corresponding cell's supply and demand from the minimum value obtained in the previous step is:

Supply = 50 – 50 = 0

Demand = 225 – 50 = 175

As the supply is 0, we need to allocate 50 to that cell and strike the corresponding column and then move down to the next cell, i.e., the cell with the value 18.

Step 5: For the cell with value 18, the minimum of supply and demand is min(300, 175) = 175.

Step 6: The difference between the corresponding cell's supply and demand from the minimum value obtained in the previous step is:

Supply = 300 – 175 = 125

Demand = 175 – 175 = 0

As demand is 0, we need to allocate 175 to that cell and strike the corresponding column and then move right to the next cell, i.e., the cell with the value 14.

Step 7: For the cell with value 14, the minimum of supply and demand is min(125, 275) = 125.

Step 8: The difference between the corresponding cell's supply and demand from the minimum value obtained in the previous step is:

Supply = 125 – 125 = 0

Demand = 275 – 125 = 150

As the supply is 0, we need to allocate 125 to that cell and strike the corresponding column and then move down to the next cell, i.e., the cell with the value 13.

Step 9: For the cell with value 13, the minimum of supply and demand is min(400, 150) = 150.

Step 10: The difference between the corresponding cell's supply and demand from the minimum value obtained in the previous step is:

Supply = 400 – 150 = 250

Demand = 150 – 150 = 0

As demand is 0, we need to allocate 125 to that cell and then move right to the next cell, i.e., the cell with the value 10. Here, we don't get any further cells to strike off.

Now, we should calculate the total minimum cost using the allocated values and the corresponding cell values.

Here, the transportation path is:

O1 → D1, O1 → D2, O2 → D2, O2 → D3, O3 → D3, O3 → D4

Therefore, the total cost = (200 × 11) + (50 × 13) + (175 × 18) + (125 × 14) + (150 × 13) + (250 × 10)

= 2200 + 650 + 3150 + 1750 + 1950 + 2500

= Rs. 12,200

2.3 Least Cost Method

This method is also known as the Matrix Minimum method or Inspection method. It starts by making the first allocation to the cell for which the transportation cost per unit is lowest. The row or column for which the capacity is exhausted or requirement is satisfied is removed from the transportation table. We follow the procedure with the reduced matrix until all the requirements are satisfied. If there is a tie for the lowest cost cell while making any allocation, the choice may be made for a row or a column by which maximum requirement is exhausted. If there is a tie in making this allocation as well, then we can arbitrarily choose a cell for allocation.

Least Cost Method

Step1: Select the cell having lowest unit cost in the entire table and allocate the minimum of supply or demand values in that cell.

Step2: Then eliminate the row or column in which supply or demand is exhausted. If both the supply and demand values are same, either of the row or column can be eliminated.

In case, the smallest unit cost is not unique, then select the cell where maximum allocation can be made.

Step3: Repeat the process with next lowest unit cost and continue until the entire available supply at various sources and demand at various destinations is satisfied.

The method can be easily explained with the help of the following example.

Illustration 1: Find the basic feasible solution of the transportation problem of Example 4.3.1.1. by using the Least Cost method.

Solution. Here, the least cost is 37 in the cell (A, G). The requirement of the Warehouse G is 220 and the capacity of Factory A is 160. Hence, the maximum number of units that can be allocated to this cell is

160. Thus, Factory A is exhausted. The requirement of Warehouse G is reduced by 160.

Now, the least cost is 38, which is in the cell (C, E). The requirement of the Warehouse E is 90 and the capacity of Factory C is 190. Hence, the maximum number of units that can be allocated to this cell is 90. Moreover, we reduce the capacity of factory C by 90.

The least cost in the matrix is 39, which is in the cell (C, D). The requirement of Warehouse D is 80 and the capacity of Factory C is 100.Hence, the maximum number of units that can be allocated to this cell is 80. The requirement of Warehouse D is exhausted. The capacity of the Factory C is also reduced by 80.

The least cost in this matrix is 40 which is in the cell (C, F). The requirement of Warehouse F is 110 and the capacity of Factory C is 20. Hence, the maximum number of units that can be allocated to this cell is 20. Thus, Factory C is exhausted. The requirement of Warehouse F is reduced by 20. It is now 90 in the reduced matrix. The least cost is 51 in the cell (B, G) and the requirement of warehouse G is 60 units. So, we allocate 60 units to cell (B, G) and the remaining 90 units to the cell

(B, F). Thus, the allocations given using Least Cost method are as shown in the following matrix along with the cost per unit of transportation:

Factory \ Warehouse	D	E	F	G	Capacity
A	42 [80]	48 [80]	38	37	160
B	40	49 [10]	52 [11]	51 [30]	150
C	39	38	40	43 [190]	190
Requirement	80	90	110	220	500

Thus, the total transportation cost = 37× 160 + 52 × 90 + 51 × 80 + 38 × 90 + 40 × 20

= 21000

Note: This method has reduced the total transportation cost in comparison to the NWC rule.

Illustration 2: Find the basic feasible solution of the following problem using the Least Cost method:

Origin/ Distribution Centre	1	2	3	4	5	6	Availability
1	4	6	9	2	7	8	10
2	3	5	4	8	10	0	12
3	2	6	9	8	4	13	4
4	4	4	5	9	3	6	18
5	9	8	7	3	2	14	20
Requirements	8	8	16	3	8	21	

Answer. Using Least Cost method, the allocations are to be made as under:

4 units to cell (1,1), 3 units to cell (1,4), 3 units to cell (1,6), 12 units to cell (2,6), 4 units to cell (3,1), 8 units to cell (4,2), 10 units to cell (4,3), 6 units to cell (5,3), 8 units to cell (5,5) and 6 units to cell (5,6). The transportation cost = 278.

2.4 Vogel's Approximation Method (VAM)

We describe the step by step procedure for finding the initial basic feasible solution by Vogel's Approximation method (Penalty method) in the following steps:

(i) In the transportation table calculate penalties for each row (column), by taking the difference between the least and second least costs in the same row (column). We display it to the right (below) of that row (column) in a new column (row) formed by extending the table on the right (bottom). The new column and row formed by extending the table at the right and bottom are labelled as penalty column and penalty row, respectively. The differences noted in the penalty row or penalty column indicates the penalty or extra cost. If two cells in a row (or column) contain the same least costs then the difference is taken as zero.

(ii) Select the row or column with the largest penalty (largest difference) and allocate the maximum possible units to the least cost cell in the selected column or row. If there is a tie in the values of penalties, the choice may be made for that row or column, which has the least cost. In case there is a tie in such least cost as well, choice may be made from that there is a tie in such least cost as well, choice may be made from that row or column by which maximum requirements are exhausted

.The cell so chosen is allocated the units and the corresponding exhausted row or column is removed or ignored from further consideration.

(iii) Now, we determine the column and row differences for the reduced transportation table and repeat the procedure until all column and row totals are exhausted.

This method is also known as the penalty method**.**

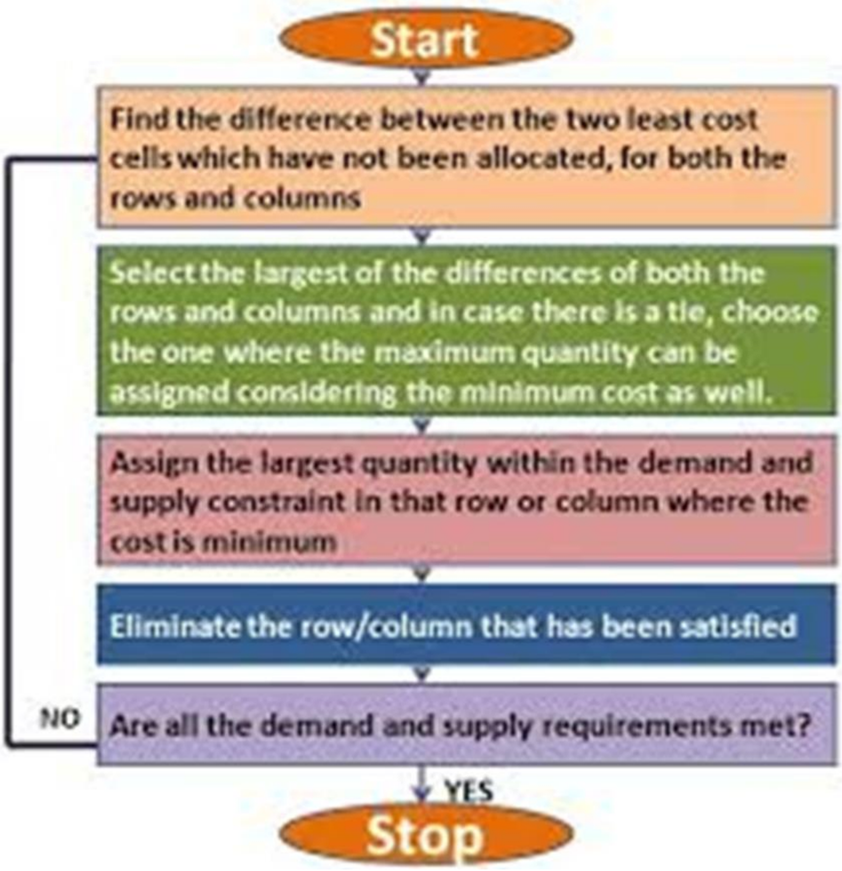


Illustration 1: Apply the Vogel's Approximation Method for finding the Basic Feasible Solution for the transportation problem of Example

Solution. In the first row, the least and the second least costs are 37 and 38 and their difference is 1. We write 1 in a new column created on the right. It is labelled Penalty.Similarly, the differences between the least and the second least costs in the second and third row, respectively, are 49 – 40 = 9 and 39 – 38 = 1. So, we write the values (differences), i.e., 9 and 1 in the penalty column.

Next, we find the differences of the least and second least elements of each of the columns D, E, F and G. These are 40-39 = 1, 48-38 = 10, 40-38 = 2 and 43-37 = 6, respectively. We write them in a newly created penalty row at the bottom of the table.

We now select the largest of these differences in the penalty row and column, which are 10 in this case. This value (10) corresponds to the second column (Column E) and the least cost in the column is 38. Hence the allocation of 90 units (the maximum requirement of warehouse E) is to be made in the cell (C, E) from Factory C. Since the column corresponding to E is exhausted, it is removed for the next reduced matrix and the capacity of C is reduced by 90.

We now take the differences between the least and the Second least cost for each row and column of the reduced matrix. In the first row, the least and the second least costs are 37 and 38 and their differences is.

1. We write it in the newly created penalty column. Similarly, we write the second difference element 51- 40 = 11 and third difference element 40-39 = 1 in the second and third row of this column. Likewise, the differences of the smallest and second smallest elements of each of the columns D, F, and G are 40–39 = 1, 4–38 = 2 and 43–37 = 6, respectively. We write these in a newly created penalty row at the bottom of the table.

Now, we select the largest of these differences in the penalty row and column, which is 11 in this case. This value (11) corresponds to Row B. Since the least cost in row is 40, we allocate 80 units (the maximum requirement of Warehouse D) to the cell (B, D). Thus, the requirement of Warehouse D is exhausted and we can remove it. We also reduce the capacity of Factory B by 80 in the next reduced matrix.

Again, in the first row, the least and the second least costs are 37 and 38 and their difference is 1. We write it to the right of this row in the newly created penalty column. Similarly, the second and third elements in the second and third rows of this column are 52–51=1 and

43–40 = 3, respectively. We write these in a newly created penalty row at the bottom of the table. Now, we select the largest of these differences, which are 6 in this case. It corresponds to Column G and the least cost in this column is 37. Hence, we allocate 160 units (the maximum capacity of Factory A) to the cell (A, G). Since Row A is exhausted, it is removed for the next reduced matrix. We also reduce the requirement of Warehouse G by 160 units.

Once again, the difference of the least costs in the first row is 52 – 51 = 1. We write it in the newly created penalty column. Similarly, for the second row, the difference is 43-40 = 3. Likewise, the differences of the least and second least elements of each of the columns F and G are 52 – 40 = 12 and 51 – 43 = 8, respectively. We write them in the newly created penalty row at the bottom of the table. The largest of these differences is 12 in this case. It corresponds to Column F and the least cost in this column is 40. Hence, we allocate 100 units from the row (the maximum capacity of Factory C) to the cell (C, F). Since Row C is exhausted, it is removed and the requirement of Warehouse F is reduced to 10 for the next reduced matrix. At the end, of the 70 units available in Factory B, we allocate 60 units to the lower cost (51), i.e., to the cell (B, G) and the remaining 10 units to the cell (B, F).

The entire procedure of allocating units by Vogel's Approximation Method is given in the following table:

Factory \ Warehouse	D	E	F	G	Capacity	$Diff_1$	$Diff_2$	$Diff_3$	$Diff_4$
A	42	48	38	**37** (160)	160	1	1	1	-
B	**40** (80)	49	**52** (10)	**51** (60)	150	9	11*	1	1
C	39	**38** (90)	**40** (100)	43	190	1	1	3	3
Requirement	80	90	110	220	500				
$Diff_1$	1	10*	2	6					
$Diff_2$	1	-	2	6					
$Diff_3$	-	-	2	6*					
$Diff_4$	-	-	12*	8					

Thus, the total transportation cost

$$= 40 \times 80 + 38 \times 90 + 52 \times 10 + 40 \times 100 + 37 \times 160 + 51 \times 60$$

$$= 3200 + 3420 + 520 + 4000 + 5920 + 306 = 20120$$

Note: The total transportation cost obtained above is the lowest total transportation cost among the three methods. Clearly the solution obtained by VAM is nearest to the optimal solution.

Illustration 2: Find the basic feasible solution of the following problem using Vogel's Approximation method:

Origin /Distribution Centre	1	2	3	4	5	6	Availability
1	4	6	9	2	7	8	10
2	3	5	4	8	10	0	12
3	2	6	9	8	4	13	4
4	4	4	5	9	3	6	18
5	9	8	7	3	2	14	20
Requirement	8	8	16	3	8	21	

Answer. The total transportation cost= 242.

2.5 Row Minima Method

In the row minima method, the first row that is the lowest cost cell is exhausted. Our aim will be to allocate the maximum either at the first source or demand at the destinations or to satisfy both. This process must be continued for all the other reduced transportation costs until and unless the supply and demand are satisfied.

In this problem, we begin with the lowest cost i.e. Rs. 2000 present at the first cell AX of the first row. So we allocate the lowest out of 1000 and 2200, so it is 1000. This deletes the first row and exhausts the supply capacity of factory A. now comes the next allocation in the next row, row 2, and cell BX. Similar as before we choose the minimum value between 1500 and 1300. So it is 1300. Hence meets the demand requirements of centre X and deleting the column 1.

Distribution centres

		X	Y	Supply
	A	Rs.2000	Rs.5380	
		1000		1000
Factories	B	Rs. 2500	Rs.2700	
		1300	200	
	C	Rs.2550	Rs.1700	
			1200	1200
	Demand	2300	1400	3700

Similarly we move to row 3 where the minimum cost is Rs. 1700 in the cell CY. Then we allocate the minimum out from 1400 and 1200. As now we see that the demand of the distribution centers is 1400 and we could allocate only 1200 so we add more 200 in the cell BY. This satisfies the column Y and hence crossed. The similar thing we do with row 2. In row 2 the supply was satisfied by 1300+200=1500 and similarly crossed out. Same with row3.

Therefore Z = Rs. (2000 × 1000 + 2500 × 1300 + 2700 200 + 1700 × 1200 = Rs. 78, 30.000.

2.6 column minima method

In the column minima method, we begin with the first column and allocate gradually moving towards the lowest cost cell of the column. This system is continued until the first destination centre is satisfied or the capacity of the second is exhausted, or both happens. So there are three cases which are as follows:

1. You can cross the first column and move towards the right column if the demand of the first distribution centre is satisfied.
2. If the supply of the ith factoryis fulfilled, then the ith row must be crossed out, and the first column must be considered as the remaining demand.
3. Now the last case, if the demand for the distribution centresand the ith factories are satisfied, then it is said to make a zero allotment in the second lowest row of the cost cell of the first column. So in that case the ith row and the columns are crossed off to move to the second column.

This process is continued until all the reduced transportation table conditions are satisfied. The matrix below explains the whole row minima method.

Distribution centers

		X	Y	Supply
	A	Rs. 2000	Rs.5380	
		1000		1000
Factories	B	Rs. 2500	Rs. 2700	
		1300	200	1500
	C	Rs. 2500	Rs. 1700	
			1200	1200
	Demand	2300	1400	3700

So let's now start solving in this mentioned method. So here we see the lowest cost cell in the column AX. We allocate the minimum amount which is 1000 out of 2300, 1000. With this, the capacity of the factory A is exhausted and the row one is crossed out. The next allocation is done with the cell BX. Similarly, the taking the minimum amount of the first column is 2500; we allocate with the minimum 1300 in this cell. This again satisfies the distribution centre X and hence the first column is also crossed.

Now moving to the second column with the minimum cost cell CY. We allocate 1200 in this cell among 1400 and 1200. Now considering the next one BY where we can allot, and all the conditions are satisfied.

Transportation cost associated with this solution is

Z = Rs. (2000 x 1000 + 2500 x 1300 + 1700 x 1200 + 2700 x 200)

= Rs. 78.34, 000

Which is same as a result obtained by row minima method.

Let Us Sum Up

The methods of computing transportation problems were discussed in earlier sections.

2.2 Check Your Progress

1. What is the primary criterion for allocation in the Least Cost Method?

- A) Allocating to the top-left corner cell
- B) Allocating to the cell with the highest supply
- C) Allocating to the cell with the lowest cost
- D) Allocating to the cell with the highest demand

2. In the Least Cost Method, what happens after the initial allocation is made?

- A) The supply and demand are reduced and the process repeats
- B) The next allocation is made to the top-left corner
- C) All costs are recalculated from scratch
- D) The problem is solved optimally

4. What does Vogel's Approximation Method (VAM) aim to minimize?

- A) Total demand
- B) Penalty costs
- C) Total transportation cost
- D) Total supply

5. In VAM, what is the penalty cost?

- A) The difference between the highest and lowest costs in a row or column
- B) The sum of all costs in a row or column
- C) The highest cost in a row or column
- D) The lowest cost in a row or column

6. What is the primary criterion for allocation in the Least Cost Method?

- A) Allocating to the top-left corner cell
- B) Allocating to the cell with the highest supply
- C) Allocating to the cell with the lowest cost
- D) Allocating to the cell with the highest demand

7. In the Least Cost Method, what happens after the initial allocation is made?

- A) The supply and demand are reduced and the process repeats
- B) The next allocation is made to the top-left corner
- C) All costs are recalculated from scratch
- D) The problem is solved optimally

UNIT SUMMARY

In this chapter, we have introduced transportation problems. As one can express these problems in terms of LPPs so transportation problems is also considered as one of the sub – classes of Linear Programming Problems and the objective of the problem is to determine the optimal amount to ship/ transport from each origin to destination. Here, we have explained how to formulate mathematically a transportation problem, finding initial basic feasible solution, performing optimality test and moving towards optimal solution.

GLOSSARY

Destination- It is the location to which shipments are transported.

Unit Transportation cost- It is the cost of transporting one unit of the consignment from an origin to a destination.

Perturbation Technique-It is a method used for modifying a degenerate transportation problem, so that the degeneracy can be resolved.

Feasible Solution-A solution that satisfies the row and column sum restrictions and also the non-negativity restrictions is a feasible solution.

Basic Feasible Solution-A feasible solution of (m X n) transportation problem is said to be basic feasible solution, when the total number of allocations is equal to (m + n – 1).

Optimal Solution- A feasible solution is said to be optimal solution when the total transportation cost will be the minimum cost.

In the sections that follow, we will concentrate on algorithms for finding solutions to transportation problems.

Self-Assessment Questions

1. Explain transportation problem. Is this a linear programming problem?

2. Explain method for obtaining initial basic feasible solution of Transportation problem:

(i) The North-West Corner Rule,

(ii) Vogel's Approximation Method (VAM),

(iii) The least Cost Method.

3.

Factories (Supply): A (20 units), B (30 units), C (50 units)

Warehouses (Demand): X (40 units), Y (35 units), Z (25 units)

Transportation Costs (per unit):

From A to X: $2, A to Y: $3, A to Z: $1

From B to X: $4, B to Y: $2, B to Z: $5

From C to X: $3, C to Y: $4, C to Z: $2

Use the North-West Corner Rule to determine the initial feasible solution.

4.

Suppliers (Supply): P (50 units), Q (60 units), R (40 units)

Distribution Centers (Demand): D1 (30 units), D2 (70 units), D3 (50 units)

Transportation Costs (per unit):

From P to D1: $6, P to D2: $4, P to D3: $3

From Q to D1: $8, Q to D2: $5, Q to D3: $4

From R to D1: $3, R to D2: $6, R to D3: $5

Use the North-West Corner Rule to determine the initial feasible solution.

5.

Suppliers (Supply): P (50 units), Q (60 units), R (40 units)

Distribution Centers (Demand): D1 (30 units), D2 (70 units), D3 (50 units)

Transportation Costs (per unit):

From P to D1: $6, P to D2: $4, P to D3: $3

From Q to D1: $8, Q to D2: $5, Q to D3: $4

From R to D1: $3, R to D2: $6, R to D3: $5

Use the North-West Corner Rule to determine the initial feasible solution.

Exercise

6.

suppliers (Supply): P (50 units), Q (60 units), R (40 units)

Distribution Centers (Demand): D1 (30 units), D2 (70 units), D3 (50 units)

Transportation Costs (per unit):

From P to D1: $6, P to D2: $4, P to D3: $3

From Q to D1: $8, Q to D2: $5, Q to D3: $4

From R to D1: $3, R to D2: $6, R to D3:

Use the North-West Corner Rule to determine the initial feasible solution.

7.

Factories (Supply): A (20 units), B (30 units), C (50 units)

Warehouses (Demand): X (40 units), Y (35 units), Z (25 units)

Transportation Costs (per unit):

From A to X: $2, A to Y: $3, A to Z: $1

From B to X: $4, B to Y: $2, B to Z: $5

From C to X: $3, C to Y: $4, C to Z: $2, Use Vogel's Approximation Method to determine the initial feasible solution.

8.

Suppliers (Supply): P (50 units), Q (60 units), R (40 units)

Distribution Centers (Demand): D1 (30 units), D2 (70 units), D3 (50 units)

Transportation Costs (per unit):

From P to D1: $6, P to D2: $4, P to D3: $3

From Q to D1: $8, Q to D2: $5, Q to D3: $4

From R to D1: $3, R to D2: $6, R to D3: $5, Use Vogel's Approximation Method to determine the initial feasible solution.

9.

Plants (Supply): M1 (25 units), M2 (35 units), M3 (45 units)

Stores (Demand): S1 (30 units), S2 (40 units), S3 (35 units)

Transportation Costs (per unit):

- From M1 to S1: $5, M1 to S2: $8, M1 to S3: $6
- From M2 to S1: $7, M2 to S2: $4, M2 to S3: $5

- From M3 to S1: $6, M3 to S2: $7, M3 to S3: $4, Use Vogel's Approximation Method to determine the initial feasible solution.

10.

Warehouses (Supply): W1 (40 units), W2 (30 units), W3 (50 units)

Stores (Demand): T1 (20 units), T2 (60 units), T3 (40 units)

Transportation Costs (per unit):

- From W1 to T1: $4, W1 to T2: $5, W1 to T3: $7
- From W2 to T1: $3, W2 to T2: $6, W2 to T3: $4
- From W3 to T1: $5, W3 to T2: $4, W3 to T3: $3

Use Vogel's Approximation Method to determine the initial feasible solution.

11.

Factories (Supply): F1 (35 units), F2 (45 units), F3 (40 units)

Distribution Centers (Demand): C1 (25 units), C2 (50 units), C3 (45 units)

Transportation Costs (per unit):

From F1 to C1: $6, F1 to C2: $5, F1 to C3: $4

From F2 to C1: $3, F2 to C2: $7, F2 to C3: $6

From F3 to C1: $5, F3 to C2: $4, F3 to C3: $5

Use Vogel's Approximation Method to determine the initial feasible solution.

12.

Factories (Supply): A (20 units), B (30 units), C (50 units)

Warehouses (Demand): X (40 units), Y (35 units), Z (25 units)

Transportation Costs (per unit):

From A to X: $2, A to Y: $3, A to Z: $1

From B to X: $4, B to Y: $2, B to Z: $5

From C to X: $3, C to Y: $4, C to Z: $2

Use the Least Cost Method to determine the initial feasible solution.

13.

Suppliers (Supply): P (50 units), Q (60 units), R (40 units)

Distribution Centers (Demand): D1 (30 units), D2 (70 units), D3 (50 units)

Transportation Costs (per unit):

From P to D1: $6, P to D2: $4, P to D3: $3

From Q to D1: $8, Q to D2: $5, Q to D3: $4

From R to D1: $3, R to D2: $6, R to D3: $5

Use the Least Cost Method to determine the initial feasible solution.

14.

Plants (Supply): M1 (25 units), M2 (35 units), M3 (45 units)

Stores (Demand): S1 (30 units), S2 (40 units), S3 (35 units)

Transportation Costs (per unit):

From M1 to S1: $5, M1 to S2: $8, M1 to S3: $6

From M2 to S1: $7, M2 to S2: $4, M2 to S3: $5

From M3 to S1: $6, M3 to S2: $7, M3 to S3: $4. Use the Least Cost Method to determine the initial feasible solution.

15.

Warehouses (Supply): W1 (40 units), W2 (30 units), W3 (50 units)

Stores (Demand): T1 (20 units), T2 (60 units), T3 (40 units)

Transportation Costs (per unit):

From W1 to T1: $4, W1 to T2: $5, W1 to T3: $7

From W2 to T1: $3, W2 to T2: $6, W2 to T3: $4

From W3 to T1: $5, W3 to T2: $4, W3 to T3: $3

Use the Least Cost Method to determine the initial feasible solution.

16.

Factories (Supply): F1 (35 units), F2 (45 units), F3 (40 units)

Distribution Centers (Demand): C1 (25 units), C2 (50 units), C3 (45 units)

Transportation Costs (per unit):

From F1 to C1: $6, F1 to C2: $5, F1 to C3: $4

From F2 to C1: $3, F2 to C2: $7, F2 to C3: $6

From F3 to C1: $5, F3 to C2: $4, F3 to C3: $5

Use the Least Cost Method to determine the initial feasible solution.

17. Find the initial basic feasible solution of the following transportation problem.

BYJU'S The Learning App

From	D_1	D_2	D_3	D_4	Supply
O_1	19	20	50	10	700
O_2	70	30	40	60	900
O_3	40	8	70	20	1800
Demand	500	800	700	1400	

18. Find the initial feasible solution of the given transportation problem so that the total cost of transportation is minimum.

BYJU'S The Learning App

From	To				Supply
	C_1	C_2	C_3	C_4	
P_1	8	6	10	9	35
P_2	9	12	13	7	50
P_3	14	9	16	5	40
Demand	45	20	30	30	

19. Obtain the initial basic feasible solution for the below transportation problem.

BYJU'S The Learning App

From	To				Supply
	D_1	D_2	D_3	D_4	
O_1	6	4	1	5	140
O_2	8	9	2	7	160
O_3	4	3	6	2	50
Demand	60	100	150	40	

20. What is North-west corner rule?

21. Why do we use the North-west corner rule?

22. Can we apply the North-west corner rule for an unbalanced transportation problem?

Answers for Check Your Progress

2.1

1. A) Northwest Corner Rule

2. B) A situation where there are more basic variables than required

3. C) Minimizing the total transportation cost

4. C) Total supply equals total demand

5. B) MODI (Modified Distribution) Method

2.2

1. B) Allocate to the cell in the top-left corner
2. A) Right then down
3. C) Total transportation cost
4. A) The difference between the highest and lowest costs in a row or column
5. C) Allocating to the cell with the lowest cost
6. A) The supply and demand are reduced and the process repeats

UNIT III

Unit introduction

The assignment problem is a fundamental issue in operations research and combinatorial optimization, focusing on the optimal allocation of resources to tasks. It is characterized by the goal of minimizing the total cost or maximizing the total efficiency of assigning a set of agents to a set of tasks, where each agent is to be assigned exactly one task and each task is to be assigned to exactly one agent. Commonly represented as a cost matrix, where the rows represent agents and the columns represent tasks, the problem seeks to determine the optimal one-to-one assignment that minimizes the overall cost or maximizes the total benefit. This problem has widespread applications in various fields, including scheduling, resource allocation, and logistics, making it crucial for enhancing operational efficiency and decision-making processes. The assignment problem can be effectively solved using algorithms such as the Hungarian method, which ensures a polynomial-time solution, thus providing a practical approach for real-world applications.

3.1 ASSIGNMENT PROBLEM

Assignment problem is a special type/case of transportation problem and hence is of linear programming problem which deals with the allocation of the various resources to the various activities on one to one basis. It does it in such a manner that the cost or time involved in the process is minimum and profit or sale is maximum. Which we already know, but here we discuss another method namely Hungarian method for solving an assignment problem. Hungarian method is shorter and easier than stepping stone and MODI methods which we have discussed in previous chapter. In this chapter, we shall explain the assignment problems including travelling salesman problem and apply Hungarian method for solving these problems.

An assignment problem may be considered as a special type of transportation problem in which there are as many jobs/sources as the number of machines/destinations so that the jobs can be assigned to machines in a one-to-one way only. The capacity of each source as well as the requirement of each destination is taken as 1. The main difference between an assignment problem and transportation problem is that in the case of assignment problem, the given matrix must necessarily be a square matrix which is not the condition for a transportation problem.

Let there be n persons and n jobs and let Cij represents the amount of time taken by ith person to complete the jth job then our objective is assignment of jobs on one-to-one basis in such a way that the total cost is minimum. The assignment problem can be stated in the form of an n×n matrix of real numbers called the cost matrix as given in the following table

Person Job

1 2 …j … n
2
.
i
.
n $C_{11} C_{12} \ldots C_{1j} \ldots C_{1n}$
$C_{21} C_{22} \ldots C_{2j} \ldots C_{2n}$
.
.
$C_{i1} C_{i2} \ldots C_{ij} \ldots C_{in}$
.
$C_{n1} C_{n2} \ldots C_{nj} \ldots C_{nn}$

3.1.1 Hungarian Method

Hungarian method is also known as Reduced Matrix Method, it is an efficient method for solving assignment problems. Hungarian method is developed by Hungarian mathematician D. Konig. The step by step procedure for obtaining an optimal solution of an assignment problem are as follows:

1. Develop the cost table from the given problem then check whether the given matrix is square i.e. is number of sources/machines is equal to the number of destinations/jobs. If not, make it square by adding a suitable number of dummy row (or column) with 0 cost/time element.
2. Locate the smallest cost element in each row of the given cost matrix and then subtract the smallest element of each column from every element of that column.
3. In the resulting cost matrix, locate the smallest element in each column and subtract the smallest element of each column from every element of that column.
4. In the modified matrix, search for an optimal assignment as follows:

a)Examine the row successively until a row with exactly one zero is found. Draw a rectangle around this zero like 0 and cross out all other zeroes in the corresponding column. Proceed in this way until all the row have been considered. If there is more than one zero in any row, don't touch that row, pass on to the next row.

b)Repeat step (a) above for the columns of the resulting cost matrix.

c)If a row or column of the reduced matrix contains more than one zeroes, arbitrarily choose a row or column having the minimum number of zeroes. Arbitrarily select any zero in the row or column so chosen. Draw a rectangle around it and cross out all the zeroes in the corresponding row and column. Repeat steps (a), (b), and (c) until all the zeroes have either been assigned (by drawing a rectangle around them) or crossed.

d)If each row or column of the resulting matrix has one and only one assigned zero, i.e. number of assigned zeroes are equal to the number of rows/columns, then the optimum assignment is made in the cells corresponding to 0 and the optimum solution of the problem is attained and we can stop here.

Otherwise, go to the next step.

5 .Draw the minimum number of horizontal and/or vertical lines through all the zeroes as follows:

i. Mark (√) the rows in which assignment has not been made.

ii. Mark (√) column, that have zeroes in the marked rows.

iii. Mark (√) rows(not already marked) which have assignments in marked columns. Then mark (√) columns, which have zeroes in newly marked rows, if any. Mark (√) rows (not already marked), which have assignments in these newly marked columns.

6.Revise the cost matrix as follows:

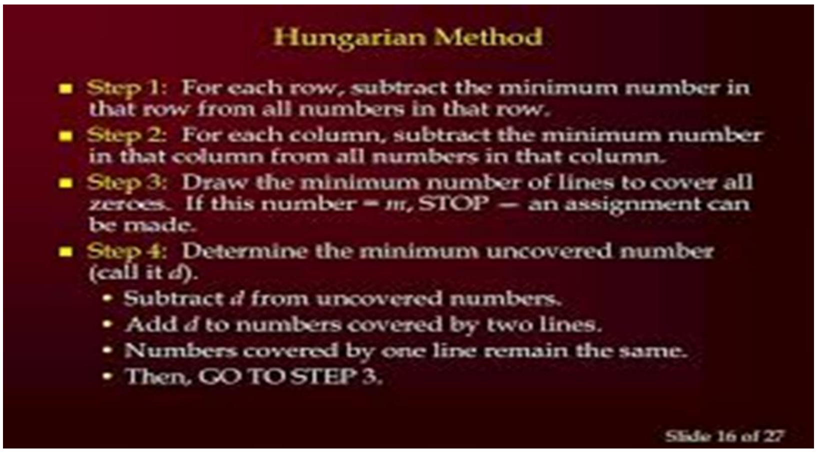
i. Find the smallest elements not covered by any of the lines.

ii. Subtract this from all the uncovered elements and add it to the elements at the intersection of the two lines.

iii. Other elements covered by the lines remain unchanged.

7.Repeat the procedure until an optimal solution attained.

Note: By drawing lines through all the unmarked rows and marked columns, we will get the required minimum number of lines.



The following example illustrate the method.

Illustration 1:

A computer centre has four expert programmers & needs to develop four application programmes. The head of the computer Centre, estimates the computer time (in minutes) required by the respective experts to develop the application programmes as follows:

PROGRAMMES / PROGRAMMERS	A	B	C	D
1	120	100	80	90
2	80	90	110	70
3	110	140	120	100
4	90	90	80	90

Find the assignment pattern that minimize the time required to develop the application programmes.

Solution:

Let us subtract the minimum element of each row from every element of that row. Note that the minimum element in the first row is 80. So, 80 is subtracted from every element of the first row. Similarly, we obtain the elements of the other rows of the resulting matrix. Thus, the modified matrix is:

	A	B	C	D
1	40	20	0	10
2	10	20	40	0
3	10	40	20	0
4	10	10	0	10

Let us now subtract the minimum element of each column from every element of that column in the resulting matrix. The minimum element in the first column is 10. So, 10 is to be subtracted from every element of the first column. Similarly, we obtain the elements of the other columns of the resulting matrix.

Thus, the resulting matrix is:

	A	B	C	D
1	30	10	0	10
2	0	10	40	0
3	0	30	20	0
4	0	0	0	10

Now starting from the first row onward, we draw a rectangle around the zero in each row having a single zero and cross all other zeros in the corresponding column. Here, in the very first row we find a single zero. So, we draw a rectangle around it and cross all the other zeroes in the corresponding column. We get

	A	B	C	D	
1	30	10		0	10
2	0	10	40	0	
3	0	30	20	0	
4	0	0	0	10	

In the second, third and fourth row, there is no single zero. Hence, we move column – wise. In the second column, we have a single zero. Hence, we draw a rectangle around it and cross all other zeroes in the corresponding row.

In the matrix above, there is no row or column, which has a single zero. Therefore, we first move row – wise to locate the row having more than one zero. The second row has two zeroes. So, we draw a rectangle arbitrarily around one of these zeroes and cross the other one. Let us draw a rectangle around the zero in the cell (2, A) and cross the zero in cell (2, D). We cross out the other zeroes in the first column. Note that we could just as well have selected zero in the cell (2, D), drawn a rectangle around it and crossed all other zeroes. This would have led to an alternative solution.

In this way, we are left with only one zero in every row and column around which a rectangle has been drawn. This means that we have assigned only one operation to one operator. Thus, we get the optimum solution as follows:

Note that the assignment of jobs should be made on the basis of the cells corresponding to the zeroes around which rectangles have been drawn. Therefore, the optimum solution for this problem is:

1 → C, 2 → A, 3 → D, 4 → B

This means that programmer 1 is assigned programme C, programmer 2 is assigned programme A, and so on. The minimum time taken in developing the programmes is

= 80 + 80 + 100 + 90 = 350 min.

Illustration 2:

A company is producing a singl product and selling it through five agencies situated in the different cities. All of a sudden, there is a demand for the product in five more cities that do not have any agency of the company. The company is faced with the problem of deciding on how to assign the existing agencies to dispatch the product to the additional cities in such a way that the travelling distance is minimized. The distances (in km) between the surplus and the deficit cities are given in the following distance matrix

Deficit city / Surplus city	I	II	III	IV	V
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

Determine the optimum assignment schedule.

Solution:

Subtracting the minimum element of each row from every element of that row and then subtracting the minimum element of each column from every element of that column, we have

	I	II	III	IV	V
A	30	0	35	30	15
B	15	0	0	10	0
C	30	0	35	30	20
D	0	0	20	0	5
E	20	0	25	15	15

We now assign zeroes by drawing rectangles around them as explained in above example. Thus, we get

	I	II	III	IV	V
A	30	0	35	30	15
B	15	0	0	10	0
C	30	0	35	30	20
D	0	0	20	0	5
E	20	0	25	15	15

Since the number of assignments is less than number of rows (or columns), we proceed from step 5 onwards of the Hungarian method described as follows:

I. We mark (√) the rows in which the assignment has not been made . These are the 3rd& 5th row.

ii. We mark (√) the columns which have zeroes in the marked rows . This is the 2nd column.

iii. We mark (√) the rows which have assignments in marked columns . This is the 1st row.

iv. Again we mark (√) the columns which have zeroes in the newly marked row . This is the 2nd column (which has been already marked).

Deficit city / Surplus city	I	II	III	IV	V
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

Determine the optimum assignment schedule.

Solution: Subtracting the minimum element of each row from every element of that row and then subtracting the minimum element of each column from every element of that column, we have

	I	**II**	**III**	**IV**	**V**
A	30	0	35	30	15
B	15	0	0	10	0
C	30	0	35	30	20
D	0	0	20	0	5
E	20	0	25	15	15

We proceed as follows, as explained in the step 6 of the Hungarian method:

1) We find the smallest element in the matrix not covered by any of the lines. It is 15 in this case.

2) We subtract the number '15' from all uncovered elements and add it to the elements at the intersection of the two lines.

3) Other elements covered by the lines remain unchanged. Thus, we have

	I	II	III	IV	V
A	15	0	20	15	0
B	15	15	0	10	0

C	15	0	20	15	5
D	0	15	20	0	5
E	5	0	10	0	0

We repeat steps 1 to 4 of the Hungarian method and obtain the following matrix

Since each row and each column of this matrix has one and only one assigned 0, we obtain the optimum assignment schedule as follows:

A → V, B → III, C → II, D → I, E→ IV

Thus, the minimum distance is 200+130+110+50+80 = 570 km.

A solicitor's firm employs typists on an hourly piece – rate basis for their daily work. There are five typists for service and their charges and speeds are different. According to the contract, only one job is given to one typist. Find the least cost allocation for the following data:

	P	Q	R	S	T
A	85	75	65	85	75
B	90	180	66	90	78
C	75	66	57	75	69
D	80	72	60	80	72
E	76	64	56	72	68

3.1.2 UNBALANCED ASSIGNMENT PROBLEM

Some assignment problems may be unbalanced, i.e. the number of machines may be different from the number of jobs. In this case, in the obtained matrix the number of rows is not equal to the number of columns and the problem said to be an unbalanced Assignment problem. Such a problem is handled by introducing dummy row(s) if the number of rows is less than the number of columns and dummy column(s) if the number of columns is less than the number of rows. All the elements of such a dummy row/column are taken as zero. After creating dummy rows or columns, we get a balanced assignment problem and now we solved it by Hungarian method.

The following example will make the procedure clear.

Illustration:

To stimulate interest and provide an atmosphere for intellectual discussion, the faculty of mathematical sciences in an institute decides to hold special seminars four contemporary topics - Statistics, Operations Research, Discrete Mathematics, Matrices. Each such seminar is to be held once a week. However, scheduling these seminars (one for each topic and not more than one seminar per a day) has to be done carefully so that the numbers of students unable to attend is kept to a minimum. A careful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows:

	Statistics	**Operations Research**	**Discrete mathematics**	**Matrices**
Monday	50	40	60	20
Tuesday	40	30	40	30
Wednesday	60	20	30	20
Thursday	30	30	20	30
Friday	10	20	10	30

Find an optimal schedule for the seminars. Also fine the number of students who will be missing at least one seminar.

Solution:

Here the number of rows is 5 and the number of columns is 4. Therefore, the given assignment problem is unbalanced. As the number of columns is one less than the number of rows, we introduce one dummy column to convert the given assignment problem into a balanced problem. The number of students in each cell of this column is taken as zero. Thus, the problem takes the following form:

	Statistics	Operations Research	Discrete mathematics	Matrices	Dummy
Monday	50	40	60	20	0

Tuesday	40	30	40	30	0
Wednesday	60	20	30	20	0
Thursday	30	30	20	30	0
Friday	10	20	10	30	0

Now, on applying the Hungarian method (Steps 1 to 4), we get

	Statistics	Operations Research	Discrete mathematics	Matrices	Dummy
Monday	40	20	50	0	0
Tuesday	30	10	30	10	0
Wednesday	50	0	20	0	0
Thursday	20	10	10	10	0
Friday	0	0	0	10	0

Since the number of assigned zeroes < number of rows, we apply Step 5 of the Hungarian method and draw the minimum number of horizontal/ vertical lines that cover all the zeros as shown in the following table:

	Statistics	Operations Research	Discrete mathematics	Matrices	Dummy
Monday	40	20	50	0	0
Tuesday	30	10	30	10	0
Wednesday	50	0	20	0	0
Thursday	20	10	10	10	0
Friday	0	0	0	10	0

We select the minimum element from amongst the uncovered elements, which in 10 in this case. We subtract this element, i.e., 10 from each uncovered element and add it to the elements which lie at the intersection of the horizontal/vertical lines. Other covered elements will remain unaltered. Thus, the resulting matrix is:

40	20	50	0	10
20	0	20	0	0

50	0	20	0	10
10	0	0	0	0
0	0	0	10	10

Now on applying the Hungarian method, we have

40	20 ✕	50	0 ✕	10
20	0	20	0 ✕	[0]
50	[0]	20	0	10
10	0 ✕	[0]	0	0
[0]	0	0 ✕	10	10

Since each row and each column of the matrix has one and only one assigned 0, optimum assignment is made in the cells containing those zeroes around which rectangles have been drawn as Monday → Matrices, Wednesday → Operations Research, Thursday → Discrete Mathematics, Friday → Statistics The total number of students who will be missing at least one seminar = 20 + 20 + 20 + 10 = 70

Let Us Sum Up

Studying the assignment problem include understanding how to optimally allocate resources to tasks in a cost-effective or efficient manner. It involves mastering techniques such as the Hungarian method to solve the problem efficiently. Through this problem, one learns to represent and analyze cost matrices, ensuring that each agent is assigned to one task while minimizing total costs or maximizing benefits.

3.1 Check Your Progress

1. Which of the following methods is commonly used to solve the assignment problem?

- A) Simplex Method
- B) Hungarian Method
- C) North-West Corner Rule
- D) Vogel's Approximation Method

2. In an assignment problem, what does each element in the cost matrix represent?

- A) The total cost of all assignments
- B) The cost of assigning a specific agent to a specific task
- C) The total demand for each task
- D) The supply available from each agent

3. What is the objective of solving an assignment problem?

- A) Maximizing the total number of assignments
- B) Minimizing the total cost or maximizing the total efficiency of the assignments
- C) Balancing supply and demand
- D) Allocating resources to multiple tasks simultaneously

4. Which condition must be satisfied for the assignment problem to be considered balanced?

- A) The number of agents must equal the number of tasks
- B) The total cost must be minimized
- C) All agents must be assigned at least one task
- D) Each task must have multiple agents assigned to it

5.In the context of the assignment problem, what does it mean if the problem is unbalanced?

- A) The cost matrix contains negative values
- B) The number of agents does not equal the number of tasks
- C) The total cost exceeds a predefined threshold
- D) There are multiple optimal solutions

3.2 CASE OF MAXIMIZATION OF AN ASSIGNMENT PROBLEM

All problems dealt with so far were all cost-minimizing problems but assignment problem exists with profit maximization problem also. For example, profits (or anything else like revenues), which need maximization may be given in the cells instead of costs/ times. The method of solving such problems is a simple modification of the method of solving cost-minimizing assignment problems. To solve such a problem, we find the opportunity loss matrix by subtracting the value of each cell from the largest value chosen from amongst all the given cells. When the value of a cell is subtracted from the highest value, it gives the loss of amount caused by not getting the opportunity which would have given the highest value. The matrix so obtained is handled in the same way as the minimization problem. The following example illustrate the method.

Illustration 1:

Five salesmen are to be assigned to five districts. Estimates of sales revenue (in thousands) for each salesman are given as following:

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	27
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment pattern that maximizes the sales revenue.

Solution:

Since we are to maximize the sales revenue, we need to convert it into minimization form before applying the Hungarian method. For this, we obtain the opportunity loss matrix by subtracting every element in the given table from the largest element is 41. Thus, we obtain the following opportunity loss matrix:

9	3	1	13	1
1	17	13	20	5
0	14	8	11	4
19	3	0	5	5
12	8	1	6	2

Now, we apply the Hungarian method (Steps 1 to 4) and finally obtain the following result matrix:

	A	B	C	D	E
1	8	0	0	7	0
2	0	14	12	14	4
3	0	12	8	6	4
4	19	1	0	0	5
5	11	5	0	0	1

Since the number of assigned zero is less than the number of rows, we apply Step 5 of the Hungarian method and draw the minimum number of horizontal/vertical lines that cover all the zeroes as shown in the following table

[illegible]

	A	B	C	D	E	
1	[illegible]	0	0	7	0	
2	0	14	12	14	4	√
3	0	12	8	6	4	√
4	1	1	0	0	[illegible]	
5	1	5	0	0	1	
	√					

Let us now, select the minimum element from amongst the uncovered elements, which is 4 in the case. We subtract the element 4 from each of the uncovered elements and add it to the elements which lie at the intersection of the horizontal and vertical lines. Other covered elements remain unaltered. Then applying the Hungarian method to the resulting matrix. We get

	A	B	C	D	E
1	12	0	0	7	0
2	0	10	8	10	0
3	0	8	4	2	0
4	23	1	0	0	5
5	15	5	0	0	1

Since the number of assigned zeroes is equal to the number of rows, the optimum assignment has been attained and is given as

1 → B, 2 → A, 3 → E, 4 → C, 5 → D

Thus, the maximum sales revenue = 38+ 40 +37 +41 +35 thousand rupees

= 191 thousand rupees.

3.3 SEQUENCING PROBLEM

The selection of an appropriate order for finite number of different jobs to be done on a finite number of machines is called sequencing problem. In a sequencing problem we have to determine the optimal order (sequence) of performing the jobs in such a way so that the total time (cost) is minimized. Suppose n jobs are to be processed on m machines for successful completion of a project. Such type of problems frequently occurs in big industries. The sequencing problem is to determine the order (sequence) of jobs to be executed on different machines so that the total cost (time) involved is minimum.

Before developing the algorithm we define certain terms as Mij = processing time required by ith job on the jth machine (i = 1 to n, j = 1 to m)Tij = idle time on machine j from the completion of (i - 1)th job to the start of ith job.

T = elapsed time (including idle time) for the completion of all the jobs.

The problem is to determine a sequence i1, i2, ... , in, where i1, i2, ... , in is a some permutation of the integers 1,2, ... ,n that minimizes the total elapsed time T. Each job is processed on machine M1 and then on machine M2 , and we say jobs functioning order is M1M2. Before developing the algorithm in the next section we make certain assumptions.

(i) No Machine can process more than one job at a time.

(ii) Each job once started on a machine must be completed before the start of new job.

(iii) Processing times Mij's are independent of the order of processing the jobs.

(iv) Processing times Mij's are known in advance and do not change during operation.

(v) The time required in transferring a job from one machine to other machine is negligible.

The Sequencing Problem

A problem occurs when people go through three or more checkpoints at about the same time. While the printer printings the first log entry, the other print call waits. After the first log entry is printed, there is no guarantee which log entry will be printed next. Log entries may not be printed in the same order they were sent to the printer.

3.3.1Processing of n jobs through two machines:

The simplest possible sequencing problem is that of n job two machine sequencing problem in which we want to determine the sequence in which n-job should be processed through two machines so as to minimize the total elapsed time T. The problem can be described as:

a) Only two machines A and B are involved;

b) Each job is processed in the order AB.

c) The exact or expected processing times A1,A2,A3, --- , An ; B1,B2,B3, --- , Bn are known

and are provided in the following table.

Machine	Job(s)								
	1	2	3	--	-	i	- -	-	n
A	A_1	A_2	A_3	--	-	A_i	--	-	A_n
B	B_1	B_2	B_3	--	-	B_i	--	-	B_n

The problem is to find the sequence (or order) of jobs so as to minimize the total elapsed time T. The solution of the above problem is also known as Johnson's procedure which involves the following steps:

Step 1. Select the smallest processing time occurring in the list A1,A2,A3, --- , An ;B1,B2,B3, --- , Bn if there is a tie, either of the smallest processing times can be selected.

Step 2. If the least processing time is Ar , select the r th job first.If it is Bs, do the s th job last as the given order is AB

Step 3. There are now (n-1) jobs left to be ordered. Repeat steps I and II for the remaining set of processing times obtained by deleting the processing time for both the machines corresponding to the job already assigned.

Step 4. Continue in the same manner till the entire jobs have been ordered. The resulting ordering will minimize the total elapsed time T and is called the optimal sequence.

Step 5. After finding the optimal sequence as stated above find the total elapsed time and idle times on machines A and B as under:

Total elapsed time = The time between starting the first job in the optima sequence on machine A and completing the last job in he optimal machine B. Idle time on machineA = (Time when the last job in the optimal sequence on sequences is completed on machine B)- (Time when the last job in the optimal sequences is completed on machine A)

Idletimeon machine B = (Time when the first job in the optimal sequences is completed on machine A)+ on machine Q). Now since the corresponding processing time of task E on machine P is less than the corresponding processing time of task G on machine Q therefore task E will be processed in the last and task G next to last. The situation will be dealt as

Problem:
There are nine jobs, each of which must go through two machines P and Q in the order PQ, the processing times (in hours) are given below:

Machine	Job(s)								
	A	B	C	D	E	F	G	H	I
P	2	5	4	9	6	8	7	5	4
Q	6	8	7	4	3	9	3	8	11

Find the sequence that minimizes the total elapsed time T. Also calculate the total idle time for the machines in this period.

Solution:
The minimum processing time on two machines is 2 which correspond to task A on machine P. This shows that task A will be preceding first. After assigning task A, we are left with 8 tasks on two machines

Machine	B	C	D	E	F	G	H	I
P	5	4	9	6	8	7	5	4
Q	8	7	4	3	9	3	8	11

Minimum processing time in this reduced problem is 3 which correspond to jobs E and G (both

A							G	E

The problem now reduces to following 6 tasks on two machines with processing time as follows:

Machine	B	C	D	F	H	I
P	5	4	9	8	5	4
Q	8	7	4	9	8	11

Here since the minimum processing time is 4 which occurs for tasks C and I on machine P and task D on machine Q. Therefore, the task C which has less processing time on P will be processed first and then task I and task D will be placed at the last i.e., 7th sequence cell. The sequence will appear as follows:

A	C	I				D	E	G

The problem now reduces to the following 3 tasks on two machines

Machine	B	F	H
P	5	8	5
Q	8	9	8

In this reduced table the minimum processing time is 5 which occurs for tasks B and H both on machine P. Now since the corresponding time of tasks B and H on machine Q are same i.e. 8. Tasks B or H may be placed arbitrarily in the 4th and 5th sequence cells. The remaining task F can then be placed in the 6th sequence cell. Thus the optimal sequences are represented as

A	I	C	B	H	F	D	E	G

(OR)

A	1	C	H	B	F	D	E	G

Further, it is also possible to calculate the minimum elapsed time corresponding to the optimal sequencing A → I → C → B → H → F → D → E → G.

Job Sequence	Machine A		Machine B	
	Time In	Time Out	Time In	Time Out
A	0	2	2	8
I	2	6	8	19
C	6	10	19	26
B	10	15	26	34
H	15	20	34	42
F	20	28	42	51
D	28	37	51	55
E	37	43	55	58
G	43	50	58	61

Hence the total elapsed time for this proposed sequence staring from job A to completion of job G is 61 hours .During this time machine P remains idle for 11 hours (from 50 hours to 61 hours)and the machine Q remains idle for 2 hours only (from 0 hour to 2 hour).

3.3.2 Processing of n Jobs through Three Machines

The type of sequencing problem can be described as follows:

a) Only three machines A, B and C are involved;

b) Each job is processed in the prescribed order ABC

c) No passing of jobs is permitted i.e. the same order over each machine is maintained.

d) The exact or expected processing times A1,A2,A3, --- , An ; B1,B2,B3, --- , Bn and

C1,C2,C3, --- , Cn are known and are denoted by the following table Our objective will be to find the optimal sequence of jobs which minimizes the total elapsed time. No general procedure is available so far for obtaining an optimal sequence in such case.

However, the Johnson's procedure can be extended to cover the special cases where either one or both of the following conditions hold:

a) The minimum processing time on machine A ≥ the maximum processing time on machine B.

b) The minimum processing time on machine C ≥ the maximum processing time on machine B.

The method is to replace the problem by an equivalent problem involving n jobs and two machines. These two fictitious machines are denoted by G and H and the corresponding time

Gi and Hi are defined by

Gi = Ai + Bi and Bi + Ci

Now this problem with prescribed ordering GH is solved by the method with n jobs through two machines, the resulting sequence will also be optimal for the original problem

Problem :

There are five jobs (namely 1,2,3,4 and 5), each of which must go through machines A, B and C in the order ABC. Processing Time (in hours) are given below:

Machine	Job(s)								
	1	2	3	--	-	i	- -	-	n
A	A_1	A_2	A_3	--	-	A_i	--	-	A_n
B	B_1	B_2	B_3	--	-	B_i	--	-	B_n
C	C_1	C_2	C_3			C_i			C_n

Find the sequence that minimum the total elapsed time required to complete the jobs.

Solution :

Here Min $A_i = 5$; $B_i = 5$ and $C_i = 3$ since the condition of Min. $A_i \geq$ Max. B_i is satisfied the given problem can be converted into five jobs and two machines problem.

Jobs	1	2	3	4	5
Machine A	5	7	6	9	5
Machine B	2	1	4	5	3
Machine C	3	7	5	6	7

The Optimal Sequence will be

2	5	4	3	1

Total elapsed Time will be

Jobs	Machine A		Machine B		Machine C	
	In	Out	In	Out	In	Out
2	0	7	7	8	8	15
5	7	12	12	15	15	22
4	12	21	21	26	26	32
3	21	27	27	31	32	37
1	27	32	32	34	37	40

Min. total elapsed time is 40 hours.

Idle time for Machine A is 8 hrs. (32-40)

Idle time for Machine B is 25 hours (0-7, 8-12, 15-21, 26-27, 31-32 and 34-40)

Idle time for Machine C is 12 hours (0-8, 22-26.)

Problems with n Jobs and m Machines

Let there be n jobs, each of which is to be processed through m machines, say M_1, M_2, --- , M_m in the order M_1, M_2, M_3, --- , M_m. Let T_{ij} be the time taken by the i^{th} machine to complete the j^{th} job. The iterative procedure of obtaining an optimal sequence is as follows:

Step I: Find (i) $\min_j (T_{1j})$, ii) $\min_j (T_{mj})$ iii) $\max_j (T_{2j}, T_{3j}, T_{4j}, \ldots, T_{(m-1)j})$ for j=1,2,---, n

Step II: Check whether a. $\min_j(T_{1j}) \geq \max_j (T_{ij})$ for i=2,3,--- ,m-1

Or

b. $\min_j(T_{mj}) \geq \max_j (T_{ij})$ for i=2,3,---,m-1

Step III: If the inequalities in Step II are not satisfied, method fails, otherwise, go to next step.

Step IV: Convert the m machine problem into two machine problem by introducing two fictitious machines G and H, such that

$$T_{Gj} = T_{1j} + T_{2j} + \cdots + T_{(m-1)j}, \text{ and } T_{Hj} = T_{2j} + T_{3j} + \cdots + T_{mj}$$

$$T_{Gj} = T_{1j} + T_{2j} + \cdots + T_{(m-1)j}, \text{ and } T_{Hj} = T_{2j} + T_{3j} + \cdots + T_{mj}$$

Determine the optimal sequence of n jobs through 2 machines by using optimal sequence algorithm.

Step V: In addition to condition given in Step IV, if $T_{ij} = T_{2j} + T_{3j} + \cdots + T_{mj} = C$ is a fixed positive constant for all i = 1, 2, 3, , n then determine the optimal sequence of n jobs and two machines M_1 and M_m in the order M_1M_m by using the optimal sequence algorithm.

Problem:

Find an optimal sequence for the following sequencing problem of four jobs and five machines when passing is not allowed, of which processing time (in hours) is given below:

Job	Machine				
	A	B	C	D	E
1	7	5	2	3	9
2	6	6	4	5	10
3	5	4	5	6	8
4	8	3	3	2	6

Also find the total elapsed time.

Solution

Here Min. $A_i = 5$, Min. $E_i = 6$

Max. $(B_i, C_i, D_i) = 6, 5, 6$ respectively

Since Min. E_i = Max. (B_i, D_i) and Min. A_i = Max. C_i satisfied therefore the problem can be converted into 4 jobs and 2 fictitious machines G and H as follows:

	Machine A		Machine B		Machine C		Machine D		Machine E	
Job	In	Out	In	Out	In	Out	In	Out	In	Out
1	0	7	7	12	12	14	14	17	17	26
3	7	12	12	16	16	21	21	27	27	35
2	12	18	18	24	24	28	28	33	35	45
4	18	26	26	29	29	32	33	35	45	51

Thus the minimum elapsed time is 51 hours.

Idle time for machine A = 25 hours(26-51)

Idle time for machine B = 33 hours(0-7,16-18,24-26,29-51)

Idle time for machine C = 37 hours(0-12,14-16,21-24,28-29,32-51)

Idle time for machine D = 35 hours (0-14,17-21,27-28,35-51)

Idle time for machine E = 18 hours (0-17,26-27)

Let Us Sum Up

Studying the transportation problem include understanding how to optimally distribute goods from multiple sources to multiple destinations while minimizing total transportation costs. Students learn to formulate and solve transportation problems using methods such as the North-West Corner Rule, Least Cost Method, and Vogel's Approximation Method to find initial feasible solutions. They also learn to check and refine solutions for optimality using methods like the MODI (Modified Distribution) method. This problem highlights the importance of balancing supply and demand, and its practical applications span logistics, supply chain management, and operations planning, ultimately enhancing efficiency and cost-effectiveness in resource allocation and distribution.

Check Your Progress

1. What is the main objective of solving a transportation problem?

- A) Maximizing the total supply
- B) Minimizing the total transportation cost
- C) Maximizing the total demand
- D) Minimizing the transportation time

2. Which of the following methods is NOT commonly used to find an initial feasible solution in a transportation problem?

- A) North-West Corner Rule
- B) Least Cost Method
- C) Vogel’s Approximation Method
- D) Simplex Method

3. In the context of the transportation problem, what does 'balancing' mean?

- A) Ensuring that the total supply is equal to the total demand
- B) Making sure that each route is used at least once
- C) Ensuring all costs are the same
- D) Minimizing the transportation time for each route

4. Which method is used to check the optimality of a transportation problem's solution?

- A) North-West Corner Rule
- B) MODI (Modified Distribution) Method
- C) Least Cost Method
- D) Critical Path Method (CPM)

5. In the North-West Corner Rule, how is the initial allocation determined?

- A) By allocating to the cell with the lowest cost
- B) By allocating to the top-left (north-west) cell of the matrix and moving right and down
- C) By calculating penalties and allocating to the highest penalty cell
- D) By balancing supply and demand in each row and column

UNIT SUMMARY

The Assignment Problem is a fundamental combinatorial optimization problem that involves finding the most efficient way to assign a set of tasks to a set of agents. The goal is to minimize the total cost or maximize the total profit of the assignments. And it is a crucial topic in operations research and optimization, with wide-ranging applications in various fields. Understanding its formulation, solving methods, and applications can significantly enhance decision-making processes in complex scenarios.

GLOSSARY

Cost Matrix - A matrix representing the cost of assigning each task to each agent. The element cijc_{ij}cij in the matrix denotes the cost of assigning task iii to agent jjj.

Decision Variables- Variables that represent the assignment choices. In the Assignment Problem, xijx_{ij}xij is a binary variable that equals 1 if task iii is assigned to agent jjj and 0 otherwise.

Hungarian Algorithm- A combinatorial optimization algorithm specifically designed to solve the Assignment Problem efficiently. It finds the optimal assignment that minimizes the total cost in polynomial time.

Linear Programming- A mathematical method for determining a way to achieve the best outcome in a given mathematical model. For the Assignment Problem, it involves formulating the problem as a linear program and solving it using methods like the Simplex algorithm.

Optimal Assignment- The assignment of tasks to agents that results in the minimum total cost (or maximum total profit). This solution satisfies all constraints and is derived through optimization techniques.

Constraints- Conditions that must be met for a solution to be feasible. In the Assignment Problem, constraints ensure that each task is assigned to exactly one agent and each agent is assigned exactly one task.

Generalized Assignment Problem (GAP)- A variation of the Assignment Problem where each agent can be assigned multiple tasks, subject to capacity constraints. The objective is still to minimize total cost or maximize total profit, but with additional complexity due to the capacity limits.

Self-Assessment Questions

1.Given: n=5n = 5n=5 jobs with the following processing times on Machine 1 and Machine 2.

Job 1: (2, 3)

Job 2: (4, 2)

Job 3: (3, 5)

Job 4: (6, 1)

Job 5: (1, 4)

2. Find the sequence of jobs that minimizes the total completion time.

Given: n=4n = 4n=4 jobs with the following processing times on Machine 1 and Machine 2.

Job 1: (1, 3)

Job 2: (2, 4)

Job 3: (3, 2)

Job 4: (4, 1)

3. Determine the optimal job sequence to minimize the makespan.

Given: n=3n = 3n=3 jobs with the following processing times on Machine 1 and Machine 2.

Job 1: (5, 6)

Job 2: (2, 8)

Job 3: (4, 3)

4. Apply Johnson's Rule to find the optimal job sequence.

Given: n=4n = 4n=4 jobs with the following processing times on Machine 1 and Machine 2.

Job 1: (3, 5)

Job 2: (7, 4)

Job 3: (2, 3)

Job 4: (6, 2)

5. Determine the sequence that minimizes idle time for both machines.

Given: n=5n = 5n=5 jobs with priorities and processing times on Machine 1 and Machine 2.

Job 1: (3, 2), Priority: 1

Job 2: (4, 3), Priority: 2

Job 3: (5, 6), Priority: 3

Job 4: (2, 4), Priority: 2

Job 5: (6, 1), Priority: 1

6. Schedule the jobs respecting their priorities to minimize the makespan.

Given: n=3n = 3n=3 jobs with due dates and processing times on Machine 1 and Machine 2.

Job 1: (4, 3), Due date: 7

Job 2: (3, 5), Due date: 10

Job 3: (2, 4), Due date: 9

7. Find the sequence that minimizes the total earliness and tardiness penalties.

Given: n=4n = 4n=4 jobs with different processing times on Machine 1 and Machine 2.

Job 1: (5, 7)

Job 2: (3, 4)

Job 3: (6, 2)

Job 4: (4, 5)

8. Determine the sequence to minimize the total completion time considering different machine capabilities.

Given: n=4n = 4n=4 jobs with the following processing times on Machine 1 and Machine 2 and precedence constraints (Job 1 must be completed before Job 2).

Job 1: (3, 5)

Job 2: (4, 6)

Job 3: (2, 4)

Job 4: (5, 3)

9. Find the optimal job sequence.

Given: n=3n = 3n=3 jobs with setup times and processing times on Machine 1 and Machine 2.

Job 1: (4, 5), Setup: (1, 2)

Job 2: (3, 6), Setup: (2, 1)

Job 3: (5, 4), Setup: (1, 3)

Exercise

10. Determine the sequence to minimize the total time including setup times.

Given: n=6n = 6n=6 jobs that can be processed in two batches on Machine 1 and Machine 2.

Jobs: (3, 5), (4, 3), (2, 4), (5, 2), (3, 3), (6, 1)

11. Find the optimal batching to minimize the makespan.

Given: n=5n = 5n=5 jobs with the following processing times on Machine 1 and Machine 2, and a maintenance window from time 5 to 6 on Machine 1.

Job 1: (2, 3)

Job 2: (3, 4)

Job 3: (4, 2)

Job 4: (1, 5)

Job 5: (5, 1)

12. Find the optimal sequence considering the maintenance window.

Given: n=4n = 4n=4 jobs with the following processing times on Machine 1 and Machine 2.

Job 1: (4, 2)

Job 2: (3, 5)

Job 3: (2, 3)

Job 4: (5, 4)

13. Determine the sequence to maximize throughput (number of jobs completed per unit time).

Given: n=3n = 3n=3 jobs with the following processing times on Machine 1 and Machine 2 and limited resources available.

Job 1: (6, 2)

Job 2: (4, 4)

Job 3: (5, 3)

14. Find the sequence that optimally uses the resources to minimize the makespan.

Given: n=5n = 5n=5 jobs with processing times on Machine 1 and Machine 2 and a maximum allowable completion time of 20 units.

Job 1: (2, 3)

Job 2: (4, 5)

Job 3: (3, 2)

Job 4: (5, 6)

Job 5: (1, 4)

15. Determine the sequence to minimize overtime.

Given: n=4n = 4n=4 jobs with the following processing times on Machine 1, Machine 2, and Machine 3.

Job 1: (2, 3, 4)

Job 2: (5, 2, 3)

Job 3: (3, 4, 2)

Job 4: (4, 5, 1)

16. Find the sequence of jobs that minimizes the total completion time.

Given: n=5n = 5n=5 jobs with the following processing times on Machine 1, Machine 2, and Machine 4.

Job 1: (4, 2, 3)

Job 2: (3, 5, 2)

Job 3: (5, 3, 4)

Job 4: (2, 4, 5)

Job 5: (3, 2, 6)

17. Determine the optimal job sequence to minimize the makespan.

Given: n=3n = 3n=3 jobs with the following processing times on Machine 1, Machine 2, and Machine 3.

Job 1: (3, 5, 2)

Job 2: (2, 6, 3)

Job 3: (4, 2, 5)

18. Apply an extension of Johnson's Rule to find the optimal job sequence for three machines.

Given: n=4n = 4n=4 jobs with the following processing times on Machine 1, Machine 2, and Machine 3.

Job 1: (5, 3, 2)

Job 2: (2, 4, 3)

Job 3: (3, 5, 4)

Job 4: (4, 2, 5)

19. Determine the sequence that minimizes idle time for all three machines.

Given: n=5n = 5n=5 jobs with priorities and processing times on Machine 1, Machine 2, and Machine 3.

Job 1: (3, 2, 4), Priority: 1

Job 2: (4, 3, 5), Priority: 2

Job 3: (5, 6, 3), Priority: 3

Job 4: (2, 4, 2), Priority: 2

Job 5: (6, 1, 5), Priority: 1

20. Schedule the jobs respecting their priorities to minimize the makespan.

Given: n=3n = 3n=3 jobs with due dates and processing times on Machine 1, Machine 2, and Machine 3.

Job 1: (4, 3, 5), Due date: 10

Job 2: (3, 5, 4), Due date: 12

Job 3: (2, 4, 6), Due date: 14

21. Find the sequence that minimizes the total earliness and tardiness penalties.

Given: n=4n = 4n=4 jobs with different processing times on Machine 1, Machine 2, and Machine 3.

Job 1: (5, 6, 7)

Job 2: (3, 4, 5)

Job 3: (6, 2, 4)

Job 4: (4, 5, 6)

21. Determine the sequence to minimize the total completion time considering different machine capabilities.

Given: n=3n = 3n=3 jobs with the following processing times on Machine 1, Machine 2, and Machine 3 and precedence constraints (Job 1 must be completed before Job 2).

Job 1: (4, 2, 3)

Job 2: (5, 3, 4)

Job 3: (2, 4, 5)

22. Find the optimal job sequence considering the precedence constraint.

Given: n=3n = 3n=3 jobs with setup times and processing times on Machine 1, Machine 2, and Machine 3.

Job 1: (4, 5, 3), Setup: (1, 2, 1)

Job 2: (3, 6, 2), Setup: (2, 1, 2)

Job 3: (5, 4, 6), Setup: (1, 3, 1)

23. Determine the sequence to minimize the total time including setup times.

Given: n=6n = 6n=6 jobs that can be processed in two batches on Machine 1, Machine 2, and Machine 3.

Jobs: (3, 5, 2), (4, 3, 5), (2, 4, 6), (5, 2, 3), (3, 3, 4), (6, 1, 5)

24. Find the optimal batching to minimize the makespan.

Given: n=4n = 4n=4 jobs and m=3m = 3m=3 machines with the following processing times:

Job 1: (3, 2, 5)

Job 2: (2, 4, 3)

Job 3: (4, 3, 2)

Job 4: (5, 1, 4)

25. Find the sequence of jobs that minimizes the total completion time.

Given: n=5n = 5n=5 jobs and m=4m = 4m=4 machines with the following processing times:

Job 1: (2, 3, 4, 1)

Job 2: (3, 5, 2, 2)

Job 3: (1, 4, 3, 5)

Job 4: (4, 2, 5, 3)

Job 5: (5, 1, 2, 4)

26. Determine the optimal job sequence to minimize the makespan.

Given: n=4n = 4n=4 jobs and m=3m = 3m=3 machines with the following processing times:

Job 1: (5, 6, 3)

Job 2: (4, 3, 7)

Job 3: (3, 5, 4)

Job 4: (2, 4, 6)

27. Apply an extension of Johnson's Rule to find the optimal job sequence for three machines.

Given: n=4n = 4n=4 jobs and m=4m = 4m=4 machines with the following processing times:

Job 1: (4, 5, 2, 3)

Job 2: (2, 3, 4, 5)

Job 3: (3, 4, 5, 2)

Job 4: (5, 2, 3, 4)

28. Determine the sequence that minimizes idle time for all four machines.

Given: n=5n = 5n=5 jobs and m=3m = 3m=3 machines with priorities and processing times:

Job 1: (3, 2, 4), Priority: 1

Job 2: (4, 3, 5), Priority: 2

Job 3: (5, 6, 3), Priority: 3

Job 4: (2, 4, 2), Priority: 2

Job 5: (6, 1, 5), Priority: 1

29. Schedule the jobs respecting their priorities to minimize the makespan.

Given: n=3n = 3n=3 jobs and m=4m = 4m=4 machines with due dates and processing times:

Job 1: (4, 3, 5, 2), Due date: 10

Job 2: (3, 5, 4, 3), Due date: 12

Job 3: (2, 4, 6, 1), Due date: 14

30. Find the sequence that minimizes the total earliness and tardiness penalties.

Given: n=4n = 4n=4 jobs and m=5m = 5m=5 machines with different processing times:

Job 1: (5, 6, 7, 4, 3)

Job 2: (3, 4, 5, 6, 2)

Job 3: (6, 2, 4, 3, 5)

Job 4: (4, 5, 6, 2, 1)

31. Determine the sequence to minimize the total completion time considering different machine capabilities.

Given: n=4n = 4n=4 jobs and m=3m = 3m=3 machines with the following processing times and precedence constraints (Job 1 must be completed before Job 2, and Job 3 must be completed before Job 4):

Job 1: (4, 2, 3)

Job 2: (5, 3, 4)

Job 3: (2, 4, 5)

Job 4: (3, 5, 2)

32. Find the optimal job sequence considering the precedence constraints.

Given: n=3n = 3n=3 jobs and m=3m = 3m=3 machines with setup times and processing times:

Job 1: (4, 5, 3), Setup: (1, 2, 1)

Job 2: (3, 6, 2), Setup: (2, 1, 2)

Job 3: (5, 4, 6), Setup: (1, 3, 1)

33. Determine the sequence to minimize the total time including setup times.

Given: n=6n = 6n=6 jobs and m=3m = 3m=3 machines that can be processed in two batches with the following processing times:

Jobs: (3, 5, 2), (4, 3, 5), (2, 4, 6), (5, 2, 3), (3, 3, 4), (6, 1, 5)

Find the optimal batching to minimize the makespan.

Check your Progress Answer

3.1

1.B) Hungarian Method

2.B) The cost of assigning a specific agent to a specific task

3.B) Minimizing the total cost or maximizing the total efficiency of the assignments

4.A) The number of agents must equal the number of tasks

5.B) The number of agents does not equal the number of tasks

3.2

1.B) Minimizing the total transportation cost

2.D) Simplex Method

3.A) Ensuring that the total supply is equal to the total demand

4.B) MODI (Modified Distribution) Method

5.B) By allocating to the top-left (north-west) cell of the matrix and moving right and down

UNIT IV

Unit Introduction

Network models are a crucial tool in operations research and management science, providing a structured way to analyze and optimize complex systems involving interconnected elements. These models represent systems as a network of nodes (points) and arcs (lines) where nodes typically represent entities such as locations, tasks, or events, and arcs represent the connections or pathways between these entities, often associated with costs, capacities, or durations. Network models are widely used to solve problems related to transportation, supply chain management, project scheduling, telecommunications, and many other fields. By applying techniques such as shortest path algorithms, maximum flow algorithms, and project evaluation and review techniques (PERT), network models help decision-makers efficiently manage resources, reduce costs, and improve operational efficiency. The visualization and analytical power of network models make them indispensable for addressing complex logistical and organizational challenges.

4.1 NETWORK MODELS

Network models are abstract representations used to describe complex systems composed of interconnected components or entities. These models are utilized in various fields such as computer science, sociology, biology, and transportation, among others. Here are a few types of network models:

Graph Theory: Graph theory is the mathematical study of graphs, which consist of vertices (nodes) connected by edges. It's a fundamental tool for analyzing network structures and properties.

Social Network Analysis (SNA): SNA examines social structures through the use of network and graph theory. It maps and measures relationships between individuals, groups, or organizations, often visualizing them as networks to understand patterns of interaction and influence.

Transportation Networks: These models represent physical networks like road systems, railways, and airline routes. They help optimize transportation routes, analyze traffic patterns, and plan infrastructure development.

Communication Networks: These models describe the flow of information between devices, such as computer networks, the internet, and telecommunications networks. They are vital for understanding data transmission, network protocols, and optimizing communication efficiency.

Biological Networks: Biological systems, such as protein interactions, neural networks, and ecosystems, can be represented as networks. These models help scientists understand the structure and function of complex biological systems.

Economic Networks: Economic networks model interactions between economic entities, such as firms, consumers, and markets. They are used to study trade patterns, supply chains, and economic resilience.

Power Grid Networks: Power grid models represent the interconnected electrical grid, including power plants, substations, and transmission lines. They are crucial for ensuring the stability and reliability of electrical systems.

4.1.1 Introduction to Project Scheduling

Some projects can be defined as a collection of inter-related activities which must be completed in a specified time according to a specified sequence and require resources, such as personnel, money, materials, facilities and so on. For instance like the projects of construction of a bridge, a highway, a power plant, repair and maintenance of an oil refinery and so on.

The growing complexities of today's projects had demanded more systematic and more effective planning techniques with the Objectives of optimizing the efficiency of executing the project. Efficiency here refers to effecting the utmost reduction in the time required to complete a project while ensuring optimum utilisation of the available resources. Project management has evolved as a new field with the development of two analytic techniques for planning, scheduling and controlling projects. These are the Critical Path Method (CPM) and the Project Evaluation and Review Technique (PERT). PERT and CPM are basically time-oriented methods in the sense that they both lead to the determination of a time schedule.

It consists of three basic phases namely:

- planning
- scheduling
- controlling

Project Planning:

In this phase following activities are performed:

- Identify various tasks or work elements to be performed in the project.
- Determine requirement of resources, such as men, materials, and machines, for carrying out activities listed above
- Estimate costs and time for various activities
- Specify the inter-relationship among various activities
- Develop a network diagram showing the sequential inter-relationships between the various activities

Project Scheduling:

Once the planning phase is over, scheduling of the project starts where each of the activities required to be performed.

Project Control:

Project control refers to comparing the actual progress against the estimated schedule. If significant differences are observed then you need to re-schedule the project to update or revise the uncompleted part of the project. Estimate the durations of activities. Take into account the resources required for these execution in the most economic manner

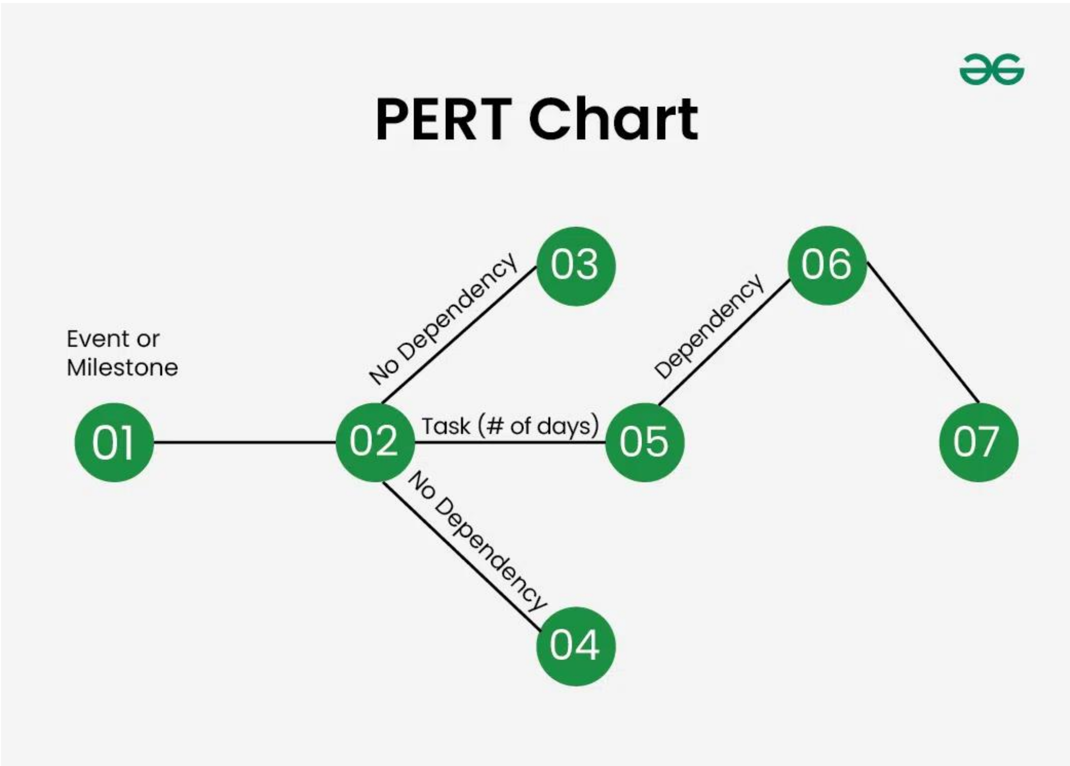
Based on the above time estimates, a time chart showing the start and finish times for each activity is prepared.Use the time chart for the following exercises:

a. To calculate the total project duration by applying network analysis techniques, such as forward (backward) pass and floats calculation.
b. To identify the critical path
c. To carry out resource smoothing (or levelling) exercises for critical or scarce resources including re-costing of the schedule taking into account resource constraints

4.2 PROGRAM EVALUATION AND REVIEW TECHNIQUE (PERT)

A PERT chart is a project management tool used to schedule, organise, and co- ordinate tasks within a project. PERT stands for (Program Evaluation Review Technique), a methodology developed by the U.S. Navy in the 1950s to manage the Polaris submarine missile program.

Program Evaluation and Review Technique (PERT) and Critical Path Method (CPM) are both project management techniques designed to assist in planning, scheduling, and controlling complex projects. PERT originated in the 1950s as a tool for managing large-scale defense projects and is characterized by its probabilistic approach, utilizing three time estimates for each activity: optimistic, pessimistic, and most likely. By calculating the expected time for each activity using these estimates, PERT helps identify the critical path, which is the longest sequence of dependent activities that determines the minimum time required to complete the project.



The PERT chart is used to schedule, organize and co-ordinate tasks within the project. the objective of PERT chart is to determine the critical path, which comprises critical activities that should be completed on schedule. This chart is prepared with the help of information generated in project planning activities such as estimation of effort, selection of suitable process model for software development and decomposition of tasks into subtasks.

4.2.1 Some key points about PERT are as follows:

- PERT was developed in connection with an R&D work. Therefore, it had to cope with the uncertainties that are associated with R&D activities. In PERT, the total project duration is regarded as a random variable. Therefore, associated probabilities are calculated so as to characterise it.
- It is an event-oriented network because in the analysis of a network, emphasis is given on the important stages of completion of a task rather than the activities required to be performed to reach a particular event or task.
- PERT is normally used for projects involving activities of non-repetitive nature in which time estimates are uncertain.
- It helps in pinpointing critical areas in a project so that necessary adjustment can be made to meet the scheduled completion date of the project.

4.2.2 Characteristics of PERT:

The main characteristics of PERT are as following :

1. It serves as a base for obtaining the important facts for implementing the decision-making.
2. It forms the basis for all the planning activities.
3. PERT helps management in deciding the best possible resource utilization method.
4. PERT take advantage by using time network analysis technique.
5. PERT presents the structure for reporting information.
6. It helps the management in identifying the essential elements for the completion of the project within time.
7. It specifies the activities that from the critical path.
8. It describes the probability of completion of project before the specified date.
9. It describes the dependencies of one or more tasks on each other.

10. It represents the project in graphical plan form.

4.2.3 Advantages of PERT.

It has the following advantages

1. Estimation of completion time of project is given by the PERT.
2. It supports the identification of the activities with slack time.
3. The start and dates of the activities of a specific project is determined.
4. It helps project manager in identifying the critical path activities.
5. PERT makes well organized diagram for the representation of large amount of data.

4.2.4 Disadvantages of PERT.

It has the following disadvantages :

1. The complexity of PERT is more which leads to the problem in implementation.
2. The estimation of activity time are subjective in PERT which is a major disadvantage.
3. Maintenance of PERT is also expensive and complex.
4. The actual distribution of may be different from the PERT beta distribution which causes wrong assumptions.
5. It under estimates the expected project completion time as there is chances that other paths can become the critical path if their related activities are deferred.

Let Us Sum Up

Program Evaluation and Review Technique (PERT) is a valuable tool in project management, designed to analyze and schedule tasks within a project to ensure timely completion. One of the key learnings from studying PERT is its ability to handle uncertainty and variability in project timelines by using probabilistic estimates for task durations. This technique allows project managers to create a realistic schedule that considers both the average time required for tasks and the potential variability, thereby minimizing the risk of delays.

Check Your Progress

1. What is the primary purpose of using PERT in project management?

- A) To determine resource allocation
- B) To estimate project costs
- C) To analyze and schedule tasks
- D) To conduct risk assessment

2.Which of the following best describes a critical path in PERT?

- A) The path with the most number of tasks
- B) The path that involves the most critical tasks
- C) The longest path in terms of duration
- D) The shortest path in terms of duration

3.What type of estimates are typically used for task durations in PERT?

- A) Fixed estimates
- B) Worst-case estimates
- C) Probabilistic estimates
- D) Average estimates

4.PERT is particularly useful for managing projects that involve:

- A) Fixed timelines and budgets
- B) Complex and uncertain environments
- C) Short-duration tasks
- D) Single-task dependencies

5.Which technique is commonly used in PERT to identify critical activities?

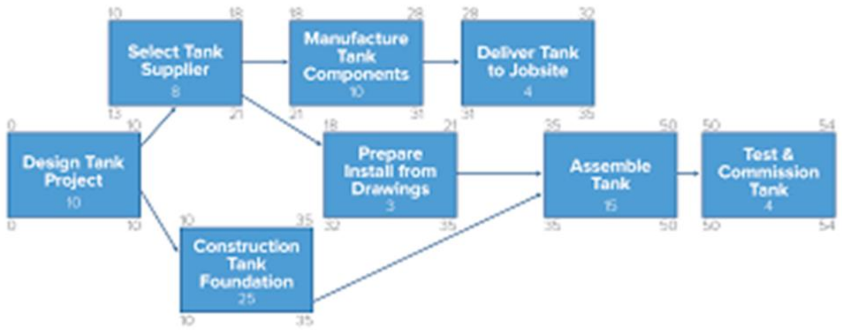
- A) Monte Carlo simulation
- B) Decision tree analysis
- C) Gantt chart
- D) Critical Path Method (CPM)

4.3 CRITICAL PATH METHOD (CPM)

Critical Path Method (CPM), developed around the same time as PERT, is more deterministic in nature and relies on a single time estimate for each activity. CPM focuses on identifying the critical path by considering the longest path of dependent activities without incorporating uncertainties. Both techniques provide valuable insights into project scheduling, resource allocation, and identifying activities that can be delayed without delaying the overall project completion time, ultimately aiding project managers in making informed decisions to ensure project success.

The Critical Path Method (CPM) is one of several related techniques for doing project planning. CPM is for projects that are made up of a number of individual "activities." If some of the activities require other activities to finish before they can start, then the project becomes a complex web of activities. Some key points about PERT are as follows:

- CPM was developed in connection with a construction project, which consisted of routine tasks whose resource requirements and duration were known with certainty. Therefore, it is basically deterministic.
- CPM is suitable for establishing a trade-off for optimum balancing between schedule time and cost of the project.
- CPM is used for projects involving activities of repetitive nature.



Finding the critical path is very helpful for project managers because it allows them to:

- Accurately estimate the total project duration.
- Estimate the time that's necessary to complete each project task.

- Identify critical activities which must be completed on time and require close supervision.
- Find out which project tasks can be delayed without affecting the project schedule by calculating slack for each task.
- Identify task dependencies, resource constraints and project risks.
- Prioritize tasks and create realistic project schedules.

4.3.1 When Should You Use Critical Path Analysis?

Critical path analysis is another way of referring to the critical path method. As noted, it's used by industries with complex projects, such as aerospace, defense, construction and product development.

Therefore, critical path analysis is a crucial first step in developing a project schedule. It's done early in the life cycle of a project, usually in the planning phase, but it's not unheard of to have CPM as part of a project proposal before the project has been approved.

By understanding which are the critical tasks in a project you can focus on getting those done if time, resources and costs are an issue. Knowing this in advance of executing a project will help you deliver that project successfully.

4.3.2 Difference between PERT and CPM

PERT (Program Evaluation and Review Technique) and CPM (Critical Path Method) are both project management tools used for scheduling and controlling projects, but they differ in several key aspects:

1. **Nature of Estimates**:
   - PERT: Utilizes probabilistic time estimates, considering three time estimates for each activity: optimistic, pessimistic, and most likely. It incorporates uncertainty into the project schedule calculations.
   - CPM: Relies on deterministic time estimates, using a single time estimate for each activity. It does not consider uncertainty in activity durations.
2. **Focus on Critical Path**:

- PERT: Identifies the critical path by considering both the sequence and duration uncertainties of activities. It accounts for variability in activity times and calculates the expected duration of the critical path.
- CPM: Determines the critical path based solely on activity sequence and deterministic activity durations. It focuses on identifying the longest path of dependent activities without considering uncertainties.

3. **Application**:
   - PERT: Initially developed for large-scale, high-uncertainty projects such as those in the defense and aerospace industries. It is particularly useful when there is considerable variability in activity durations.
   - CPM: Originally developed for construction projects, CPM is well-suited for projects with relatively low uncertainty and where activity durations can be reliably estimated.
4. **Complexity**:
   - PERT: More complex due to its probabilistic nature and the incorporation of multiple time estimates for each activity. It requires additional calculations to determine the expected duration of the critical path.
   - CPM: Relatively simpler, as it relies on deterministic time estimates for activities and straightforward calculations to identify the critical path.
5. **Risk Management**:
   - PERT: Provides a better framework for risk management by explicitly considering uncertainties in activity durations. It allows project managers to assess the likelihood of meeting project deadlines.
   - CPM: Does not directly address uncertainty, making it less suitable for projects with high levels of risk and variability.

In summary, while both PERT and CPM are valuable tools for project scheduling, they differ in their approach to time estimation, handling of uncertainty, complexity, and suitability for different types of projects. PERT is more suitable for projects with high uncertainty, while CPM is better suited for projects with relatively low variability in activity durations.

4.3.2 PERT/CPM Network Components and Precedence Relationship

PERT/CPM Network Components and Precedence Relationship PERT/CPM networks consist of two major components as discussed below:

Events: An event represents a point in time that signifies the completion of some activities and the beginning of new ones. The beginning and end points of an activity are thus described by 2 events usually known as the tail and head events. Events are commonly represented by circles (nodes) in the network diagram.

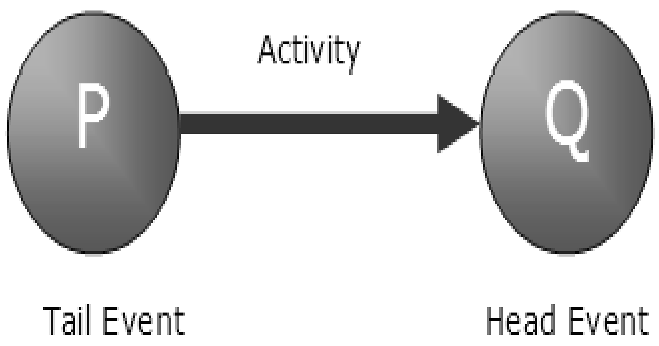
They do not consume time and resource.

Activities: Activities of the network represent project operations or tasks to be conducted. An arrow is commonly used to represent an activity, with its head indicating the direction of progress in the project. Activities originating from a certain event cannot start until the activities terminating at the same event have been completed. They consume time and resource.

Events in the network diagram are identified by numbers. Numbers are given to events such that the arrow head number is greater than the arrow tail number.

Activities are identified by the numbers of their starting (tail) event and ending (head) event.

In following Figure the arrow (P.Q) extended between two events represents the activity. The tail event P represents the start of the activity and the head event Q represents the completion of the activity

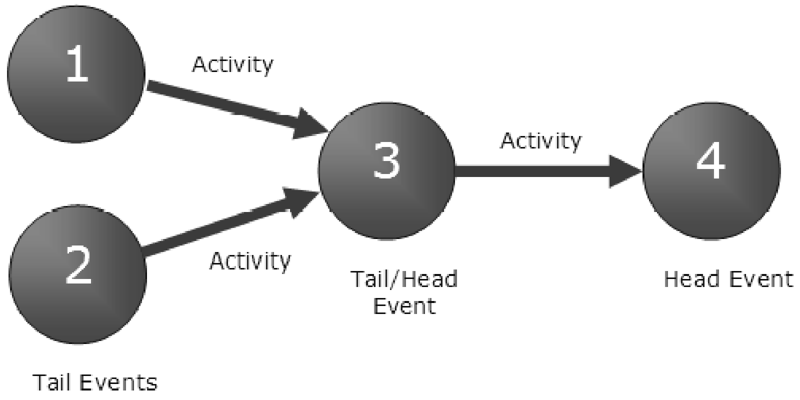


4.3.3 Basic PERT-CPM network

The following figure is example of another PERT-CPM network with activities (1, 3), (2, 3) and

(3, 4). As the figure indicates,

activities (1, 3) and (2, 3) need to be completed before activity (3,



4.4 Z PERT-CPM network

The rules for constructing the arrow diagram are as follows:

- Each activity is represented by one and only one arrow in the network
- No two activities can be identified by the same head and tail events
- To ensure the correct precedence relationship in the arrow diagram, we need to answer the following pointsas we add every activity to the network:

  What activities must be completed immediately before these activity can start?

  What activities must follow this activity?

  What activity must occur concurrently with this activity?

This rule is self-explanatory. It actually allows for checking (and rechecking) the precedence relationships as one progresses in the development of the network.

For instance: Construct the arrow diagram comprising activities A, B, C and L

such that the following relationships are satisfied:

1) A, B and C the first activities of the project, can start simultaneously.

2) A and B precede D.

3) B precedes E, F and H.

4) F and C precede G.

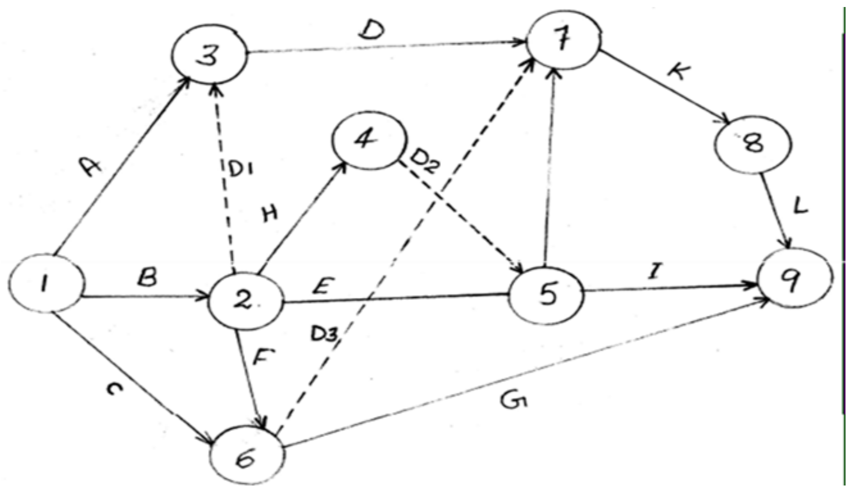
5) E and H precede I and J.

6) C, D, F and J precede K.

7) K precedes L.

8) I, G and L are the terminal activities of the project.

Solution:



Note: A dummy activity in a project network analysis has zero duration.

4.5 Critical Path Calculations

The application of PERT/CPM should ultimately yield a schedule specifying the start and completion time of each activity. The arrow diagram is the first step towards achieving that goal. The start and completion timings are calculated directly on the arrow diagrams using simple arithmetic. The end result is to classify the activities as critical or non-critical.

An activity is said to be critical if a delay in the start of the course makes a delay in the completion time of the entire project.

A non-critical activity is such that the time between its earliest start and its latest completion time is longer than its actual duration. A non-critical activity is said to have a slack or float time.

Determination of the Critical Path

A critical path defines a chain of critical activities that connects the start and end events of the arrow diagram. In other words, the critical path identifies all the critical activities of a project.

The critical path calculations are done in two phases:

The first phase is called the Forward Pass. In this phase all calculations begin from the start node and move to the end node. At each node a number is computed representing the earliest occurrence time of the corresponding event. These numbers are shown in squares. Here we note the number of heads joining the event. We take the maximum earliest timing through these heads.

The second phase is called the Backwards Pass. It begins calculations from the "end" node and moves to the "start" node. The number computed at each node is shown in a triangle near the end point, which represents the latest occurrence time of the corresponding event. In backward pass, we see the number of tails and take minimum value through these tails.

Let ESi be the earliest start time of all the activities emanating from event i. Then ESi represents the earliest occurrence time of event i.

If i = 1 is the "start" event then conventionally for the critical path calculations, ESi

= 0. Let Dij be the duration of the activity (i, j).

Then the forward pass calculations for all defined (i, j) activities with ESi=0 is given by the formula:

$$ES_i = maxi\{ES_i + D_{ij}\}$$

Therefore, to compute ESj for event j, we need to first compute ESi for the tail events of all the incoming activities (i, j).

With the computation of all ESj, the forward pass calculations are completed. The backward pass starts from the “end” event. The Objectives of the backward pass phase is to calculate LCi, the latest completion time for all the activities coming into the event i.

Thus, if i = n is the end event, LCn = ESn initiates the backward pass.

In general for any node i, we can calculate the backward pass for all defined activities using the formula:

$$LC_i = \min\{LC_j - D_{ij}\}$$

We can now identify the critical path activities using the results of the forward and backward passes. An activity (i,j) lies on the critical path if it satisfies the following conditions:

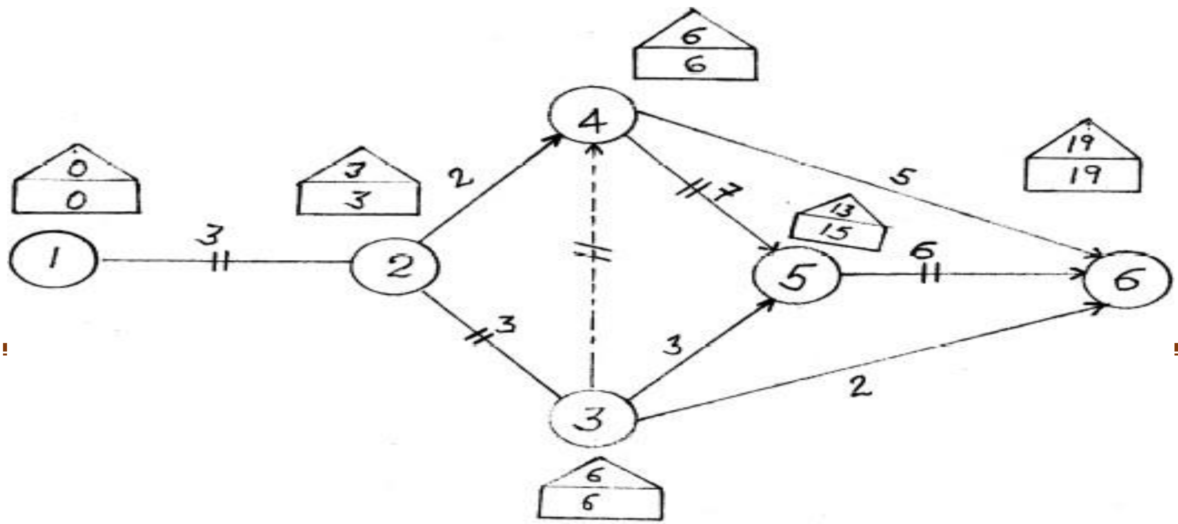
A. ESi = LCi

B. ESj = LCj

C. ESj-ESi = LCj-LCi = Dij

These conditions actually indicate that there is no float or slack time between the earliest stand and the latest start of the activity. Thus, the activity must be critical.

Thus, the activity must be critical. In the arrow diagram these are characterised by same numbers within rectangles and triangles at each of the head and tail events. The difference between the



numbers in rectangles or triangles at the head event and the number within rectangles or triangles at the tail event is equal to the duration of the activity. Thus, we will get a critical path, which is a chain of connected activities, spanning the network from start to end. For instance: Consider a network which stands from node 1 and terminates at node 6, the time required to perform each activity is indicated on the arrows.

Analysis work

Solution: Let us start with forward pass with ESi = 0.

Since there is only one incoming activity (1, 2) to event 2 with D12 = 3. ES2 = ES1+ DS2 = 0+3=3.

Let us consider the end 3, since there is only one incoming activity (2, 3) to event 3, with D23 = 3.

S3 = ES2+ D23 = 3+3 = 6.

To obtain ES4, since there are two activities A (3, 4) and (2, 4) to the event 4 with D24 = 2

and D34 = 0.

ES4= maxi=2, 3 {ESi + De4}

= max {ES2 +D24, ES3 + D34}

= max {3+2, 6+0} = 6

Similarly, ES5 = 13 and ES6 = 19. This completes the first phase. In the second phase we have

LC6= 19 = ES6

LC5 = 19-6 = 13

LC4= min J = 5, 6 {LCj – D4j} = 6 LCs = 6, LC2 = 3 and LC1 = 0

- Therefore, activities (1, 2), (2, 3) (3, 4) (4, 5) (5, 6) are critical and (2, 4) (4, 6), (3, 6), are non-critical.
- Thus, the activities (1, 2), (2, 3), (3, 4), (4, 5) and (5, 6) define the critical path which is the shortest possible time to complete the project.

4.6 DETERMINATION OF FLOATS

Following the determination of the critical path, we need to compute the floats for the non-critical activities. For the critical activities this float is zero. Before showing how floats are determined, it is necessary to define two new times that are associated with each activity. These are as follows:

- Latest Start (LS) time and
- Earliest Completion (EC) time

We can define activity (i, j) for these two types of time by $LS_{ij} = LC_j – D_{ij}$

$EC_{ij} = ES_i + D_{ij}$

There are two important types of floats namely:

- Total Float (TF)
- Free Float (FF)

The total float TF_{ij} for activity (i, j) is the difference between the maximum time available to perform the activity (= $LC_j – ES_i$) and its duration (= D_{ij})

$TF_{ij} = LC_j – ES_i – D_{ij} = LC_j – EC_{ij} = LS_{ij} – ES_i$

The free float is defined by assuming that all the activities start as early as possible. In this case FF_{ij} for activity (i,j) is the excess of available time (= $ES_i – ES_i$) over its deviation (= D_{ij}); that is, $FF_{ij} = ES_i – ES_i = D_{ij}$

Note

For critical activities float is zero. Therefore, the free float must be zero when the total float is zero. However, the converse is not true, that is, a non-critical activity may have zero free floats.

Let us consider the example taken before the critical path calculations.

Activity (i j)	Duration D_{ij}	Earliest		Latest		Table Float TF_{ij}	Free Float FF_{ij}
		Start ES_i	Completion EC_{ij}	Start LS_{ij}	Completion ΔLC_j		
(1, 2)	3	0	3	0	3	0*	0
(2, 3)	3	3	6	3	6	0*	0
(2, 4)	2	3	5	4	6	1	1
(3, 4)	0	6	6	6	6	0*	0
(3, 5)	3	6	9	10	13	4	4
(3, 6)	2	6	8	17	19	11	11
(4, 5)	7	6	13	6	13	0*	0
(4, 6)	5	6	11	14	19	8	8
(5, 6)	6	13	19	13	19	0*	0

4.6.1 Float for non-critical activities

Total float = ESij = LFij - ESij

Free float = Total float - - Head slack

For instance: A project consists of a series of tasks A, B, C, – D, – E, F, G, H, I with the following relationships:

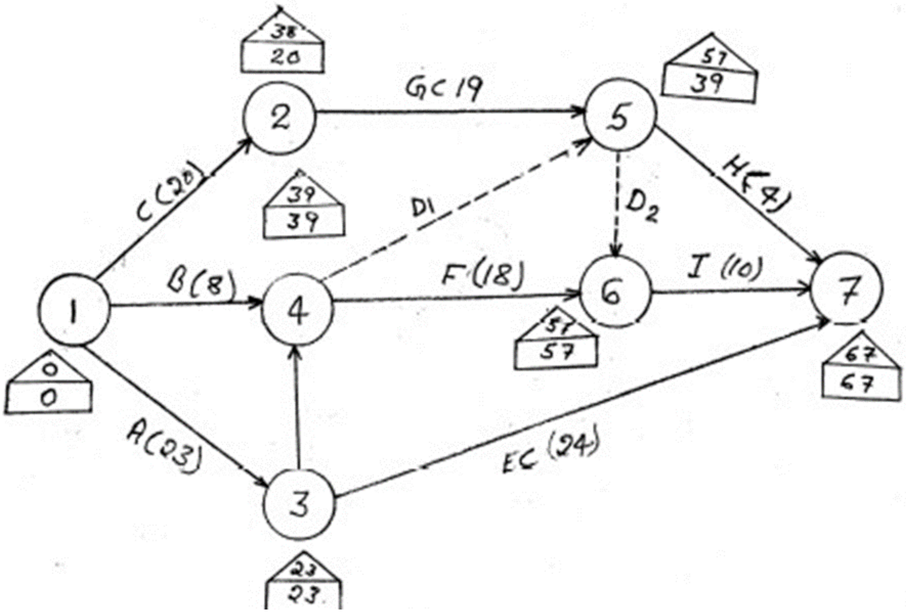
- W < X,Y means X and Y cannot starts until W is completed
- X,Y < W means W cannot start until both X and Y are completed

With this notation construct the network diagram having the following constraints A < D, E; B, D < F; C < G, B < H; F, G < I.

Also find the minimum time of completion of the project, the critical path, and the total floats of each task, when the time (in days) of completion of each task is as follows:

Task	A	B	C	D	E	F	G	H	I
Time	23	8	20	16	24	18	19	4	10

Solution:



Analysis table

ES1 = 0, ES2 = 20, ES3 = 23, ES4 = 59, ES5 = 39, ES6 = 57, ES7 = 67

Activity (i, j)	Duration D_{ij}	Earliest		Latest		Table Float TF_{ij}	Free Float FF_{ij}
		Start ES_{ij}	Finish Ee_{ij}	Start L_j-D_{ij}	Finish L_j		
(1, 2)	20	0	20	18	38	18	0
(1, 3)	23	0	23	0	23	0*	0
(1, 4)	8	0	8	31	39	31	31
(2, 5)	19	20	39	38	57	18	0
(3, 4)	16	23	39	23	39	0*	0
(3, 7)	24	23	47	43	67	20	20
(4, 5)	0	39	39	57	57	10	0
(4, 6)	18	39	57	39	57	0*	0
(5, 6)	0	39	39	57	57	18	18
(5, 7)	4	39	43	63	67	24	24
(6, 7)	10	37	67	57	67	0*	0

Activity table

Critical path is 1 – 3 – 4 – 6 – 7.

4.7 Project Management – PERT

The analysis in CPM does not take in the cases where time estimates for the different activities are probabilistic. It also does not consider explicitly the cost of schedules. Here we will consider both probability and cost aspects in project scheduling.

Probability considerations are incorporated in project scheduling by assuming that the time estimate for each activity is based on 3 different values. They are as follows:

a = the optimistic time, which will be required if the execution of the project goes extremely well. b = the pessimistic time, which will be required if everything goes bad.

m = the most likely time, which will be required if execution is normal.

The most likely estimate m need not coincide with the mid-point of a and b.

Then the expected duration of each activity D can be obtained as the mean and 2m. Therefore,

We can use this estimate to study the single estimate D in the critical path calculation.

The variance of each activity denoted by V is defined by,

The earliest expected times for the node i is denoted by E(i). For each node i, E(i) is obtained by taking the sum of expected times of all activities leading to the node i, when more than one activity leads to a node i, then the greatest of all E(i) is chosen. Let i be the earliest occurrence time of the event i, we can consider i as a random variable. Assuming that all activities of the network are statistically independent, we can calculate the mean and the variance of i as follows:

Where, k defines the activities along the largest path leading to i.

For the latest expected time, we consider the last node. Now for each path move backwards and substitute the Dij for each activity (i, j).

Thus we have,

if only one path events from j to i or if it is the minimum of {E[Lj – Dij] for all j for which the activities (i, j) is defined.

Note: The probability distribution of times for completing an event can be approximated by the normal distribution due to central limit theorem.

Since □i represents the earliest occurrence time, event will meet a certain schedule time STi (specified by an analyst) with probability

Where,

It is a common practice to compute the probability that event i will occur no later than its LCe. Such probability will represent the chance that the succeeding events will occur within the (ESe, LCe) duration.

Determine the following:

a) Expected task time and their variance

b) The earliest and latest expected times to reach each event

c) The critical path

d) The probability of an event occurring at the proposed completion data if the original contract time of completing the project is 41.5 weeks.

e) The duration of the project that will have 96% chances of being completed.

B) The E-values and L-values are shown in figure. The critical path is shown by

thick line in the figure. The critical path is 1-4-7 and the earliest completion time for the project is 42.8 weeks.

C) The last event 7 will occur only after 42.8 weeks. For this we require only the duration of critical activities. This will help us in calculating

the standard duration of the last event.

Expected length of critical path = 33 + 9.8 = 42.8 Variance of article path length = 5.429 + 0.694 = 6.123

Probability of meeting the schedule time is given by (From normal distribution table)

Thus, the probability that the project can be completed in less than or equal to 41.5 weeks is 0.30. In other words probability that the project will get delayed beyond 41.5 weeks is 0.70

D) Given that P (Z ≤ Ki) = 0.95. But Z0.9S = 1.6 u, from normal distribution table

Then 1.6 u

Sji = 1.64 X 2.47 + 42.8 = 46.85 weeks.

In PERT (Program Evaluation and Review Technique), three time estimates are used for each activity:

1. Optimistic Time (a): This is the shortest possible time required to complete an activity under ideal conditions. It represents the best-case scenario where everything goes smoothly without any delays or issues.
2. Pessimistic Time (b): This is the longest possible time required to complete an activity, considering all possible delays, setbacks, and obstacles. It represents the worst-case scenario.
3. Most Likely Time (m): This is the best estimate of the time required to complete an activity under normal conditions, taking into account typical resources, skill levels, and circumstances. It reflects the most probable duration of the activity based on past experience and expert judgment.

These three time estimates help account for uncertainty in activity durations and allow for a more realistic estimation of the project schedule. They are used to calculate the expected time for each activity, which is then used to determine the critical path and overall project duration.

Let Us Sum Up

Critical Path Method (CPM) is a vital project management technique that focuses on planning and scheduling activities within a project to ensure efficient execution and timely completion. One of the primary learnings from studying CPM is its emphasis on identifying critical activities that directly influence the project's overall duration. By mapping out a network diagram that depicts task dependencies and their respective durations, CPM helps project managers pinpoint the longest sequence of tasks, known as the critical path. Tasks on the critical path must be completed on time to prevent delays in the project's final deadline, making them a top priority for resource allocation and monitoring.

Check Your Progress

1. What is the primary objective of using CPM in project management?

- A) To estimate project costs
- B) To identify critical activities and sequences
- C) To allocate resources efficiently

- D) To monitor project progress

2. Which of the following is true about the critical path in CPM?

- A) It is the shortest path in terms of duration
- B) It is the sequence of tasks with the least resource requirements
- C) It is the longest path in terms of duration
- D) It includes all non-critical activities

3. How are activities represented in a CPM network diagram?

- A) As rectangles
- B) As circles
- C) As diamonds
- D) As triangles

4. Which technique is used to calculate the Early Start (ES) and Early Finish (EF) times of activities in CPM?

- A) Monte Carlo simulation
- B) Gantt chart analysis
- C) Backward pass calculation
- D) Forward pass calculation

5. What is the total float (slack) of an activity in CPM?

- A) The time by which an activity can be delayed without delaying the project
- B) The difference between its Early Start and Late Finish times
- C) The longest path in the network diagram
- D) The shortest path in the network diagram

UNIT SUMMARY

These models are widely used in various fields, such as transportation, telecommunications, and project management, to optimize the flow of resources, information, or tasks through a network. Key types of network models include the shortest path problem, which seeks the quickest route between two nodes; the maximum flow problem, which aims to find the greatest possible flow in a network without exceeding capacity constraints; and the minimum cost flow problem, which focuses on minimizing the cost of transporting goods through a network.

GLOSSARY

1. Node (Vertex)

A fundamental unit in a network model that represents an intersection point or entity in the network. Nodes can represent various elements such as cities in a transportation network, computers in a communication network, or tasks in a project management network.

2. Edge (Arc)

A connection between two nodes in a network model. Edges can represent routes, communication links, or dependencies and often have associated attributes such as length, capacity, or cost.

3. Shortest Path Problem

A type of network optimization problem that aims to find the minimum distance or cost path between two nodes in a network. Algorithms such as Dijkstra's or Bellman-Ford are commonly used to solve this problem.

4. Maximum Flow Problem

A network flow problem that seeks to determine the greatest possible flow from a source node to a sink node without violating capacity constraints on the edges. The Ford-Fulkerson algorithm is a typical method used to solve this problem.

5. Critical Path

In project management network models, the critical path is the longest sequence of dependent tasks that determines the shortest possible duration to complete the project. Identifying the critical path helps in pinpointing tasks that cannot be delayed without affecting the project's completion time.

Self-Assessment Questions

1.

Activity A: Optimistic time = 3 days, Most likely time = 5 days, Pessimistic time = 7 days.

Activity B: Optimistic time = 2 days, Most likely time = 4 days, Pessimistic time = 6 days.

Activity C: Optimistic time = 1 day, Most likely time = 3 days, Pessimistic time = 5 days.

Calculate the expected duration for each activity.

2.

Given the following activities with their time estimates:

Activity D: Optimistic time = 6 days, Most likely time = 8 days, Pessimistic time = 10 days.

Activity E: Optimistic time = 4 days, Most likely time = 6 days, Pessimistic time = 8 days.

Activity F: Optimistic time = 5 days, Most likely time = 7 days, Pessimistic time = 9 days. Determine the critical path and the expected project duration.

3.

Given the following activity times:

Activity G: Optimistic time = 2 days, Most likely time = 3 days, Pessimistic time = 4 days.

Activity H: Optimistic time = 3 days, Most likely time = 5 days, Pessimistic time = 7 days.

Activity I: Optimistic time = 4 days, Most likely time = 6 days, Pessimistic time = 8 days.

Determine the slack for each activity.

4.

A project consists of four activities:

Activity J: Optimistic time = 7 days, Most likely time = 9 days, Pessimistic time = 11 days.

Activity K: Optimistic time = 5 days, Most likely time = 6 days, Pessimistic time = 7 days.

Activity L: Optimistic time = 8 days, Most likely time = 10 days, Pessimistic time = 12 days.

Activity M: Optimistic time = 6 days, Most likely time = 8 days, Pessimistic time = 10 days.

Calculate the expected duration of the project and identify the critical path.

5.

Given the following activities with their durations:

- Activity A: 4 days
- Activity B: 6 days
- Activity C: 3 days
- Activity D: 5 days Determine the critical path and the total project duration.

6.

Given the following activities and their dependencies:

  - Activity E: Depends on completion of Activity F
  - Activity F: Depends on completion of both Activity G and Activity H
  - Activity G: 5 days
  - Activity H: 3 days Determine the critical path and the total project duration.

Exercise

7.

Given the following activities with their durations:

- Activity I: 4 days
- Activity J: 6 days
- Activity K: 5 days Calculate the early start, early finish, late start, and late finish times for each activity.

8.

Given the following activities with their durations:

- Activity L: 7 days
- Activity M: 9 days
- Activity N: 6 days Determine the slack for each activity.

9.

A project has the following activities with their durations and resource requirements:

- Activity O: 3 days, requires 2 workers
- Activity P: 5 days, requires 3 workers
- Activity Q: 4 days, requires 2 workers Determine the minimum number of workers required each day to complete the project in the shortest time possible.

Check Your Progress -Answers

4.1

C) To analyze and schedule tasks

C) The longest path in terms of duration

C) Probabilistic estimates

B) Complex and uncertain environments

D) Critical Path Method (CPM)

4.2.

B) To identify critical activities and sequences

C) It is the longest path in terms of duration

A) As rectangles

D) Forward pass calculation

A) The time by which an activity can be delayed without delaying the project

UNIT V

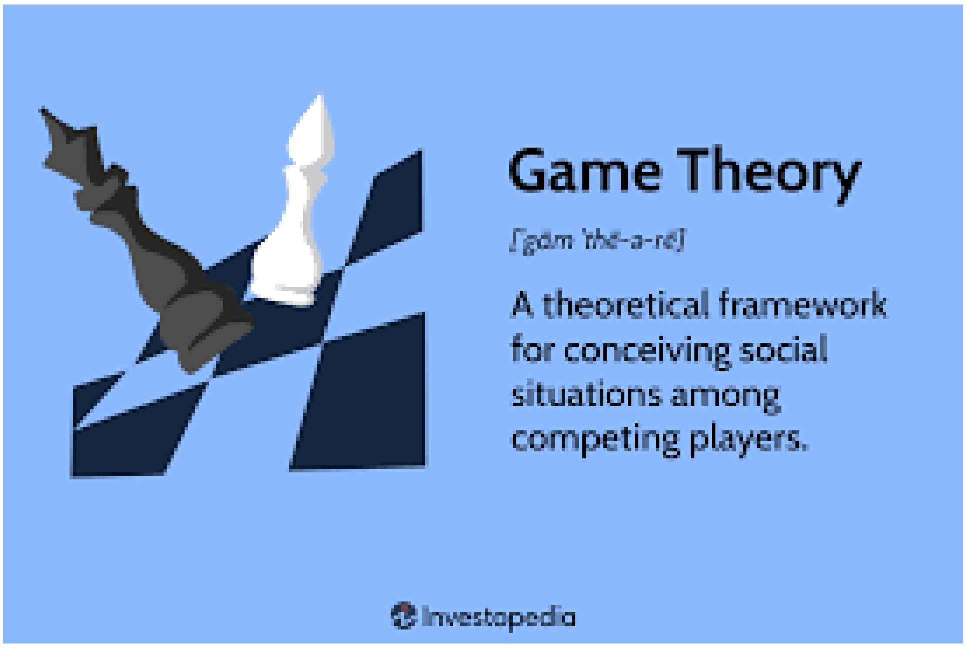
UNIT INTRODUCTION

Game theory serves as a powerful framework for analyzing strategic interactions among decision-makers. It provides insights into the behavior of rational agents in competitive or cooperative settings, offering mathematical models to predict outcomes and optimize strategies. The unit introduction to game theory explores various concepts such as players, strategies, payoffs, and equilibrium solutions like Nash equilibrium. It delves into different types of games, including simultaneous and sequential games, zero-sum and non-zero-sum games, and cooperative and non-cooperative games. Moreover, game theory extends beyond traditional economics to fields like political science, biology, and computer science, where it helps in understanding phenomena such as voting behavior, evolutionary dynamics, and algorithm design. By studying game theory, learners gain critical thinking skills to analyze complex decision-making scenarios, assess optimal strategies, and predict outcomes in competitive or cooperative environments, thereby enhancing their ability to navigate strategic interactions effectively.

5.1 GAME THEORY- AN INTRODUCTION

In real-life, we can see a great variety of competitive situations. Game theory provides tools for analyzing situations in which parties, called players, make decisions that are interdependent. This interdependence causes each player to consider the other player's possible decisions, or strategies, in formulating strategy. A solution to a game describes the optimal decisions of the players, who may have similar, opposed, or mixed interests, and the outcomes that may result from these decisions. So, one can say that it is a type of decision theory. Game theory was originally developed by John von Neumann (called the father of game theory) and his colleague Oskar Morgenstern to solve problem in economics.

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcome results. Game theory does not insist on how a game should be played but tells the procedure and principles by which action should be selected. Thus it is a decision theory useful in competitive situations. Game is defined as an activity between two or more persons according to a set of rules at the end of which each person receives some benefit or suffers loss. The set of rules defines the game. Going through the set of rules once by the participants defines a play



In this chapter, first we define some basic terms used in game theory then we shall discuss two-person zero-sum-games (also known as rectangular games), games with saddle point in which we study minimax and maximin criterion. Also, we shall explain rules of dominance which are used to reduce the size of the payoff matrix and discuss solution methods for game without saddle point namely algebraic method, graphical method and linear programming method.

The objective of these contents is to get familiar reader with game theory. After studying this chapter, reader should be able to describe the following concepts like:

- Minimax and Maximin Principle
- Pure Strategies: Game with Saddle Points
- The Rule of Dominance
- Mixed Strategies: Game without Saddle Points

5.1.1Some basic definitions

Game: A competitive situation is called a game if it has the following properties

a) There are finite numbers of participants called players.

b) Each player has finite number of strategies available to him.

c) Every game result in an outcome.

Player: Each participant of a game is called a player.

Number of players: If a game involves any two payers, it is called a two-person game. However, if the number of players is more than two, the game is known as n-person game.

Payoff: A quantitative measure of satisfaction, a person gets at the end of each play, is called a payoff.

Play: A play is said to occur when each player chooses one of his activities.

Strategy: The strategy for a payer is the list of al possible actions or moves available to him. Generally, two types of strategies are employed by players in a game.

(i)Pure strategy: It is a decision rule which is always used by the player to select any one particular course of action. The objective of the payer is to maximize gains or minimize losses.

(ii)Mixed strategy: When the players use a combination of strategies and each player always keep guessing as to which course of action is to be selected by the other on a particular occasion, then this is known as mixed strategy. Thus, the mixed strategy is a selection among pure strategies with fixed probabilities.

Zero-sum game. A game in which the algebraic sum of the outcomes for all the participants equals zero for very possible combination of strategies, is called a zero-sum game.

A game which is not zero-sum is called a non-zero-sum game.

Optimal strategy. A course of action or play which puts the player in the most preferred position, irrespective of the strategy of his competitors, is called optimal strategy.

Value of the game. The expected payoff when the players follow their optimal strategy is called the value of the game.

5.1.2 Types of Games

There are several classifications of a game. The classification may be based on various factors such as the number of participants, the gain or loss to each participant, the number of Strategies available to each participant, etc. Some of the important types of games are enumerated below.

Two person games and n-person games

In two person games, there are exactly two players and each competitor will have a finite number of strategies. If the number of players in a game exceeds two, then we refer to the game as n-person game.

Zero sum game and non-zero sum game

If the sum of the payments to all the players in a game is zero for every possible outcome of the game, then we refer to the game as a zero sum game. If the sum of the payoffs from any play of the game is either positive or negative but not zero, then the game is called a non-zero sum game

Games of perfect information and games of imperfect information

A game of perfect information is the one in which each player can find out the strategy that would be followed by his opponent. On the other hand, a game of imperfect information is the one in which no player can know in advance what strategy would be adopted by the competitor and a player has to proceed in his game with his guess works only.

Games with finite number of moves / players and games with unlimited number of moves

A game with a finite number of moves is the one in which the number of moves for each player is limited before the start of the play. On the other hand, if the game can be continued over an extended period of time and the number of moves for any player has no restriction, then we call it a game with unlimited number of moves.

Constant-sum games

If the sum of the game is not zero but the sum of the payoffs to both players in each case is constant, then we call it a constant sum game. It is possible to reduce such a game to a zero sum game.

2x2 two person game and 2xn and mx2 games

When the number of players in a game is two and each player has exactly two strategies, the game is referred to as 2x2 two person game. A game in which the first player has precisely two strategies and the second player has three or more strategies is called an 2xn game. A game in which the first player has three or more strategies and the second player has exactly two strategies is called an mx2 game.

3x3 and large games

When the number of players in a game is two and each player has exactly three strategies, we call it a 3x3 two person game. Two-person zero sum games are said to be larger if each of the two players has 3 or more choices. The examination of 3x3 and larger games is involves difficulties. For such games, the technique of linear programming can be used as a method of solution to identify the optimum strategies for the two players.

Non-constant games

Consider a game with two players. If the sum of the payoffs to the two players is not constraint in all the plays of the game, then we call it a non-constant game. Such games are divided into negotiable or cooperative games and non-negotiable or non-cooperative games.

Two-person zero sum games: A game with only two players, say player A and player B, is called a two-person zero sum game if the gain of the player A is equal to the loss of the player B, so that the total sum is zero.

Payoff matrix

When players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a payoff matrix..Since the game is zero sum, the gain of one player is equal to the loss of other and vice-versa. Suppose A has m strategies and B has n strategies.

The amount of payoff, i.e., V at an equilibrium point is known as the **value of the game**. The optimal strategies can be identified by the players in the long run.

5.1.3 Basic Assumptions of the Game

Rules of the game are given as follows.

Each player has available to him a finite number of possible courses of action. The list may not be the same for each player.

- Players act rationally and intelligently.
- The decisions of both the payers are made individually, prior to the play, with no communication between them.
- One player attempts to maximize gains and the other attempts to minimize losses.
- The players simultaneously select their respective courses of action.
- The payoff is fixed and determined in advance.
- List of strategies of each player and the amount of gain or loss on an individual's choice of

strategy is known to each player in advance.

In game theory, the maximin and minimax criteria are decision-making principles used in zero-sum games, where one player's gain is equivalent to the other player's loss. These criteria help players make strategic choices based on risk aversion or risk tolerance.

5.1.4 Properties of a Game

a) There are finite numbers of competitors called ‘players’

b) Each player has a finite number of possible courses of action called ‘strategies’

c) All the strategies and their effects are known to the players but player does not know which strategy is to be chosen.

d) A game is played when each player chooses one of his strategies. The strategies are assumed to be made simultaneously with an outcome such that no player knows his opponents strategy until he decides his own strategy.

e) The game is a combination of the strategies and in certain units which determines the gain or loss.

f) The figures shown as the outcomes of strategies in a matrix form are called ‘pay-off matrix’.

g) The player playing the game always tries to choose the best course of action which results in optimal pay off called ‘optimal strategy’.

h) The expected pay off when all the players of the game follow their optimal strategies is known as ‘value of the game’. The main objective of a problem of a game is to find the value of the game

5.1.5 Characteristics of Game Theory

a) Competitive game: A competitive situation is called a competitive game if it has the following four properties

There are finite number of competitors such that $n \geq 2$. In case $n = 2$, it is called a two person game and in case $n > 2$, it is referred as n-person game.

Each player has a list of finite number of possible activities.

A play is said to occur when each player chooses one of his activities. The choices are assumed to be made simultaneously i.e. no player knows the choice of the other until he has decided on his own.

Every combination of activities determines an outcome which results in a gain of payments to each player, provided each player is playing uncompromisingly to get as much as possible. Negative gain implies the loss of same amount.

b) Strategy: The strategy of a player is the predetermined rule by which player decides his course of action from his own list during the game.

The two types of strategy are

Pure strategy

Mixed strategy

Pure Strategy: If a player knows exactly what the other player is going to do, a deterministic situation is obtained and objective function is to maximize the gain. Therefore, the pure strategy is a decision rule always to select a particular course of action.

Mixed Strategy: If a player is guessing as to which activity is to be selected by the other on any particular occasion, a probabilistic situation is obtained and objective function is to maximize the expected gain. Thus the mixed strategy is a selection among pure strategies with fixed probabilities.

c) Number of persons: A game is called ‘n’ person game if the number of persons playing is ‘n’. The person means an individual or a group aiming at a particular objective.

Two-person, zero-sum game: A game with only two players (player A and player B) is called a ‘two-person, zero-sum game’, if the losses of one player are equivalent to the gains of the other so that the sum of their net gains is zero. Two-person, zero-sum games are also called rectangular games as these are usually represented by a payoff matrix in a rectangular form.

d) Number of activities: The activities may be finite or infinite.

e) Payoff: The quantitative measure of satisfaction a person gets at the end of each play is called a payoff.

Payoff matrix: Suppose the player A has 'm' activities and the player B has 'n' activities. Then a payoff matrix can be formed by adopting the following rules

• Row designations for each matrix are the activities available to player A

• Column designations for each matrix are the activities available to player B

• Cell entry Vij is the payment to player A in A's payoff matrix when A chooses the activity i and B chooses the activity j.

• With a zero-sum, two-person game, the cell entry in the player B's payoff matrix will be negative of the corresponding cell entry Vij in the player A's payoff matrix so that sum of payoff matrices for player A and player B is ultimately zero.

g) Value of the game: Value of the game is the maximum guaranteed game to player A (maximizing player) if both the players uses their best strategies. It is generally denoted by 'V' and it is unique.

Let Us Sum Up

Game theory provides a structured framework for analyzing and understanding strategic interactions between rational decision-makers. Key learnings from game theory include the concept of players making decisions based on their understanding of others' decisions, known as strategies. It explores various equilibrium concepts such as Nash equilibrium, where no player can improve their outcome by unilaterally changing their strategy. Game theory helps identify optimal strategies in competitive and cooperative scenarios, from simple games like Prisoner's Dilemma to complex scenarios in economics, political science, biology, and beyond. By studying game theory, individuals develop critical thinking skills to model, analyze, and predict outcomes in dynamic environments, enhancing their ability to make informed decisions in strategic situations.

Check Your Progress

1. In game theory, what does a Nash equilibrium represent?

- A) A strategy where one player always wins
- B) A strategy where each player's strategy is the best response to the others'
- C) A strategy where players cooperate to maximize their joint payoff

- D) A strategy where one player dominates all others

2.Which game is an example of a classic non-cooperative game in game theory?

- A) Chess
- B) Prisoner's Dilemma
- C) Bridge
- D) Soccer

3.In game theory, what is a dominant strategy?

- A) A strategy that always leads to a win
- B) A strategy that is the best response regardless of what the other player does
- C) A strategy that is never chosen by rational players
- D) A strategy that requires cooperation from both players

A strategy that is the best response regardless of what the other player does

4.What does the term "zero-sum game" mean in game theory?

- A) A game where the total payoffs of all players sum to zero
- B) A game with no strategies
- C) A game where one player always loses
- D) A game where one player's gain is equal to another player's loss

5.Which concept in game theory refers to a situation where both players would benefit from cooperating, but each has an incentive not to cooperate if the other does not?

- A) Nash equilibrium
- B) Prisoner's Dilemma
- C) Dominant strategy
- D) Zero-sum game\

5.2 MAXIMIN STRATEGY

Definition:

The maximin strategy is the strategy that maximizes the minimum payoff a player can receive. This is typically used by the row player (Player A) in a payoff matrix.

Meaning:

In a two-person zero-sum game, a player using a maximin strategy assumes that the opponent will play in a way that minimizes the player's payoff. Therefore, the player chooses the strategy that has the highest worst-case scenario payoff.

Steps to Determine the Maximin Strategy:

Identify the minimum payoff for each row: For each strategy (row) of Player A, find the minimum payoff that can result from any strategy chosen by Player B.

Select the maximum of these minimum payoffs: The maximin value is the highest of these minimum payoffs. The strategy corresponding to this value is the maximin strategy.

5.2.1 Maximin Criterion:

- The maximin criterion is a decision rule that focuses on minimizing the maximum possible loss.
- In a zero-sum game, the maximin strategy involves choosing the option that maximizes the minimum payoff for the player.
- Mathematically, the maximin strategy involves selecting the option with the highest minimum payoff.
- This criterion is often employed by risk-averse players who prioritize minimizing potential losses over maximizing potential gains.

5.3 MINIMAX STRATEGY

Definition:

The minimax strategy is the strategy that minimizes the maximum loss a player can suffer. This is typically used by the column player (Player B) in a payoff matrix.

Meaning:

In a two-person zero-sum game, a player using a minimax strategy assumes that the opponent will play in a way that maximizes the player's loss. Therefore, the player chooses the strategy that has the lowest worst-case scenario loss.

The maximising player lists his minimum gains from each strategy and selects the strategy which gives the maximum out of these minimum gains.

Steps to Determine the Minimax Strategy:

Identify the maximum payoff for each column: For each strategy (column) of Player B, find the maximum payoff that can result from any strategy chosen by Player A.

Select the minimum of these maximum payoffs: The minimax value is the lowest of these maximum payoffs. The strategy corresponding to this value is the minimax strategy.

For Example Consider a two person zero sum game involving the set of pure strategy for Maximising player A say Al A2 & As and for player B, B1 & B2, with the following payoff

Player B´s strategies

$$\text{Player } A\text{'s strategies} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_1 & \ldots\ldots & B_n \end{array} \\ \begin{bmatrix} a_{11} & a_1 & \ldots\ldots & a_{1n} \\ a_{21} & a_{21} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix} \end{array}$$

Suppose that player A starts the game knowing fully well that whatever strategy he adopts B will select that particular counter strategy which will minimise the payoff to A. If A selects the strategy Al then B will select B2 so that A may get minimum gain. Similarly if A chooses A2 than B will adopt the strategy of B2. Naturally A would like to maximise his maximin gain which is just the largest of row minima. Which is called 'maximin strategy'. Similarly B will minimise his minimum loss which is called 'minimax strategy'. We observe that in the above example, the maximum if row minima and minimum of column maxima are equal. In symbols. Maxi [Min.] = Mini [Max] The strategies followed by both the p1ayersarecalled ‘optimum strategy’

Value of Game. In game theory, the concept value of game is considered as very important. The value of game is the maximum guaranteed gain to the maximising player if both the players use their best strategy. It refers to the average payoff per play of the game over a period of time. Consider the following the games.

In the first game player X wins 3 points and the value of the value is three with positive sign and in the second game the player Y wins 3 points and the value of the game is -ve which indicates that Y is the Winner. The value is denoted by 'v. The different methods for solving a mixed strategy game are

- Analytical method
- Graphical method
- Dominance rule

5.3.1 MINIMAX CRITERION:

The minimax criterion is a decision rule that focuses on maximizing the minimum possible gain.

In a zero-sum game, the minimax strategy involves choosing the option that minimizes the maximum payoff for the opponent.

Mathematically, the minimax strategy involves selecting the option with the lowest maximum payoff for the opponent.

This criterion is often used by risk-tolerant players who aim to maximize their own potential gains while considering the opponent's best possible response.

Both criteria are essential in analyzing strategic interactions in games and help players formulate their optimal strategies based on their risk preferences and perceptions of the opponent's strategy. The choice between maximin and minimax strategies depends on factors such as risk aversion, strategic goals, and the level of uncertainty in the game environment.

5.4 SADDLE POINT

Definition:

A saddle point in a payoff matrix is an entry that is the minimum in its row and the maximum in its column. It represents an equilibrium point where both players' optimal strategies intersect.

Meaning:

The saddle point provides a stable outcome where neither player can improve their payoff by unilaterally changing their strategy. It represents the value of the game, and the strategies corresponding to this point are optimal for both players.

To calculate the saddle point, we need to find the minimum value in each row and the maximum value in each column. If the minimum value in a row is equal to the maximum value in its corresponding column, then that point is a saddle point.

Steps to Identify a Saddle Point:

Find the maximin value: Follow the steps to determine the maximin value and the corresponding strategy for Player A.

Find the minimax value: Follow the steps to determine the minimax value and the corresponding strategy for Player B.

Compare the maximin and minimax values: If they are equal, the corresponding entry in the payoff matrix is the saddle point.

Examples of Saddle Points

One classic example of saddle points is the game of rock-paper-scissors. In this game, each player has three possible moves: rock, paper, or scissors. Each move has a corresponding payoff, and the goal is to maximize your payoff while minimizing your opponent's. In this case, there is no saddle point as each player has equal probabilities of winning, losing, or drawing.

Another example is the game of matching pennies, where two players simultaneously choose to show either a heads or a tails of a coin. If the two outcomes match, player 1 wins, otherwise player 2 wins. In this game, there is a saddle point at (1,-1) where player 1 wins with heads and player 2 wins with tails.

5.4.1 Pure strategies: games with saddle point

Consider the payoff matrix of a game which represents payoff of player A. Now, the objective of the study is to know how these players must select their respective strategies so that they may optimize their payoff. Such a decision-making criterion is referred to as the minimax-maximin principle.

For payer A, the minimum value in each row represents the least gain to him if he chooses his particular strategy. He will then select the strategy that gives the largest gain among the row minimum values. This choice of player A is called the maximin principle and the corresponding gain is called the maximin value of the game denoted by v.

For player B, who is assumed to be loser, the maximum value in each column represents the maximum loss to value in each column represents the maximum loss to him if he chooses his particular strategy. He will then select the strategy that gives minimum loss among the column maximum values. This choice of player B is called the minimax principle and the corresponding loss is called the minimax value of the game, denoted by v .

Saddle point. A saddle point of a payoff matrix is that position in the payoff matrix where maximum of row minima coincides with the minimum of the column maxima. The saddle point need not be unique.

Value of the game. The amount of payoff at the saddle point is called the value of the game, denoted by v.

Illustration

Find Solution of game theory problem using saddle point

Player A\Player B	B1	B2	B3	B4
A1	20	15	12	35
A2	25	14	8	10
A3	40	2	10	5
A4	-5	4	11	0

Solution:

1. Saddle point testing

Players

		Player *B*			
		*B*1	*B*2	*B*3	*B*4
Player *A*	*A*1	20	15	12	35
	*A*2	25	14	8	10
	*A*3	40	2	10	5
	*A*4	-5	4	11	0

We apply the maximin (minimax) principle to analyze the game.

		Player *B*				
		*B*1	*B*2	*B*3	*B*4	Row Minimum
Player *A*	*A*1	20	15	[(12)]	35	[12]
	*A*2	25	14	8	10	8
	*A*3	40	2	10	5	2
	*A*4	-5	4	11	0	-5
	Column Maximum	40	15	(12)	35	

Select minimum from the maximum of columns

Column MiniMax = (12)

Select maximum from the minimum of rows

Row MaxiMin = [12]

Here, Column MiniMax = Row MaxiMin = 12

∴ This game has a saddle point and value of the game is 12

The optimal strategies for both players are

The player A will always adopt strategy 1

The player B will always adopt strategy 3

5.5 TWO-PERSON ZERO-SUM GAME

A game with only two-persons is said to be two-person zero-sum game if the gain of one player is equal to the loss of the other so that total sum is zero.

Payoff Matrix: In a two-person game, the payoffs in terms of gains or losses, when players select their particular strategies can be represented in the form of a matrix, called the payoff matrix of the player. If the game is zero-sum, the gain of one player is equal to the loss of the other and vice-versa. So, one player's payoff table would contain the same amounts I payoff table of the other payer with the sign changed. If the player A has strategies A1, A2, …, An and the player B has strategies B1, B2, …, Bn and if a_{ij}

represent the payoffs that the player A gains from player B when player A chooses strategy i, and player B chooses strategy j then payoff matrix for player A is given by

B chooses strategy j then payoff matrix for player A is given by

Player B's strategies

$$\text{Player } A\text{'s strategies} \quad \begin{matrix} & B_1 & B_1 & \cdots\cdots & B_n \\ \begin{matrix} A_1 \\ A_2 \\ \\ A_m \end{matrix} & \multicolumn{4}{l}{\begin{bmatrix} a_{11} & a_1 & \cdots\cdots & a_{1n} \\ a_{21} & a_{21} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}} \end{matrix}$$

5.5.1 Assumptions for two-person zero sum game

For building any model, certain reasonable assumptions are quite necessary. Some assumptions for building a model of two-person zero sum game are listed below.

- Each player has available to him a finite number of possible courses of action. Sometimes the set of courses of action may be the same for each player. Or, certain courses of action may be available to both players while each player may have certain specific courses of action which are not available to the other player.
- Player A attempts to maximize gains to himself. Player B tries to minimize losses to himself.
- The decisions of both players are made individually prior to the play with no communication between them.
- The decisions are made and announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
- Both players know the possible payoffs of themselves and their opponents. Mini max and Maxi min Principles

The selection of an optimal strategy by each player without the knowledge of the competitor's strategy is the basic problem of playing games. The objective of game theory is to know how these players must select their respective strategies, so that they may optimize their payoffs. Such a criterion of decision making is referred to as mini max-maxi min principle. This principle in games of pure strategies leads to the best possible selection of a strategy for both players.

5.5.2 The Principle of Dominance

we have discussed the method of solution of a game without a saddle point. While solving a game without a saddle point, one comes across the phenomenon of the dominance of a row over another row or a column over another column in the pay-off matrix of the game. Such a situation is discussed in the sequel.In a given pay-off matrix A, we say that the ith row dominates the kth row if

$a_{ij} \geq a_{kj}$ for all $j = 1,2,\ldots,n$ and $a_{ij} > a_{kj}$ for at least one j.

In this case, the player B will loose more by choosing the strategy for the qth column than by choosing the strategy for the pth column. So he will never use the strategy corresponding to the qth column.

When dominance of a row (or a column) in the pay-off matrix occurs, we can delete a row (or a column) from that matrix and arrive at a reduced matrix. This principle of dominance can be used in the determination of the solution for a given game.

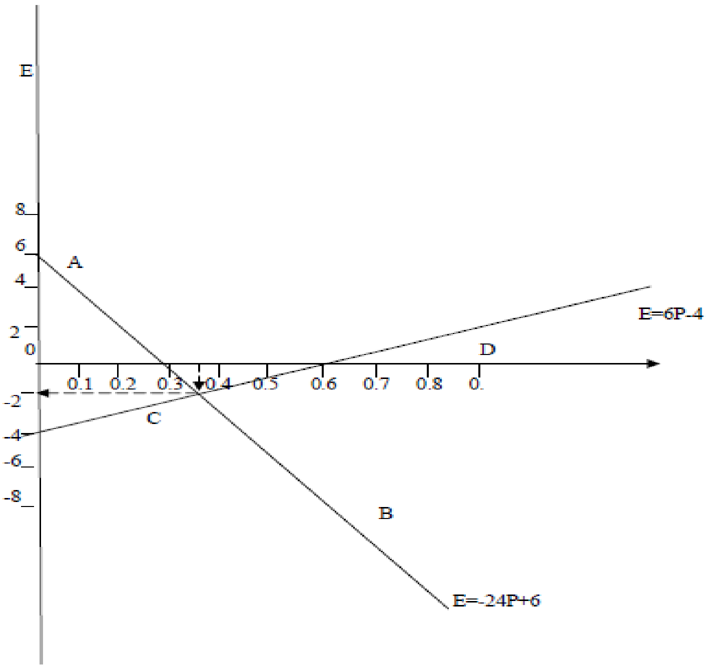
Let us consider an illustrative example involving the phenomenon of dominance in a game.

5.6 Graphical solution

Take probability and expected value along two rectangular axes in a graph sheet. Draw two straight lines given by the two equations (1) and (2). Determine the point of intersection of the two straight lines in the graph. This will provide the common solution of the two equations (1) and (2). Thus we would get the value of the game.

Represent the two equations by the two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have p = 1/ 3

and E = -2. Therefore, the value V of the game is -2.



Illustration

A company produces a basic and premium version of its product. The basic version requires 20 minutes of assembly and 15 minutes of painting. The premium version requires 30 minutes of assembly and 30 minutes of painting. If the company has staffing for 3,900 minutes of assembly and 3,300 minutes of painting each week. They sell the basic products for a profit of $30 and the premium products for a profit of \$40. How many of each version should be produced to maximize profit?

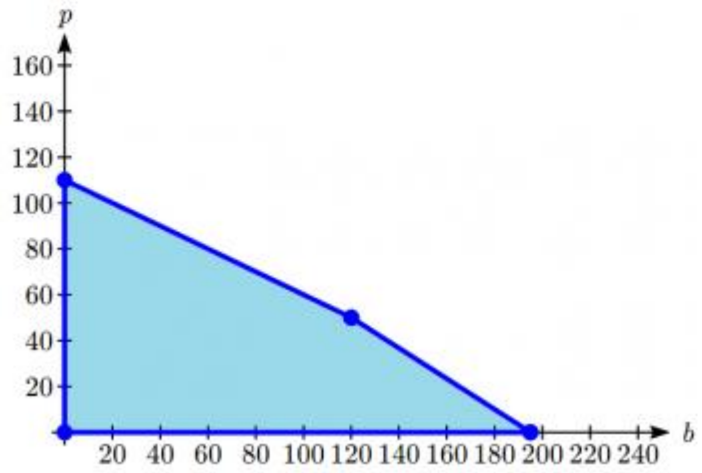
Let b = the number of basic products made, and p = the number of premium products made. Our objective function is what we're trying to maximize or minimize. In this case, we're trying to maximize profit. The total profit, P, is:

In the last section, the example developed our constraints. Together, these define our linear programming problem:

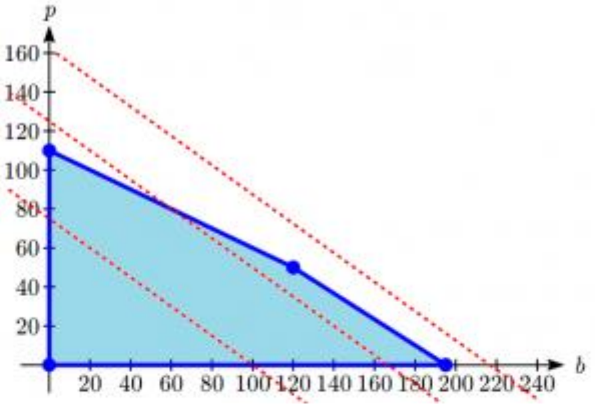
$20b + 30p \leq 3900$
$15b + 30p \leq 3300$
$b \geq 0, p \geq 0$ Objective function: MAX

Constraints: (We often say "Subject to:" or for short s.t.)

In this section, we will approach this type of problem graphically. We start by graphing the constraints to determine the feasible region – the set of possible solutions. Just showing the solution set where the four inequalities overlap, we see a clear region.

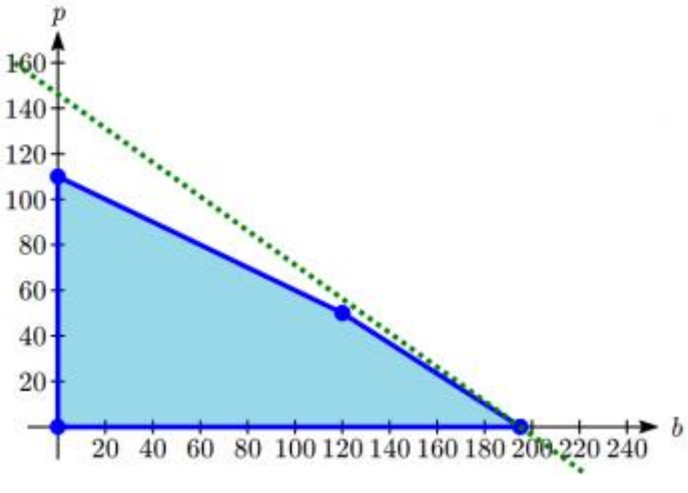


To consider how the objective function connects, suppose we considered all the possible production combinations that gave a profit of P = \$3000, so that .That set of combinations would form a line in the graph. Doing the same for a profit of \$5000 and \$6500 would give additional lines. Graphing those on top of our feasible region, we see a pattern:



Notice that all the constant-profit lines are parallel, and that in general the profit increases as we move up to upper right. Notice also that for a profit of $5000 there are some production levels inside the feasible region for that profit level, but some are outside. That means we could feasibly make $5000 profit by producing, for example, 167 basic items and no premium items, but we can't make $5000 by producing 125 premium items and no basic items because that falls outside our constraints.

The solution to our linear programming problem will be the largest possible profit that is still feasible. Graphically, that means the line furthest to the upper-right that still touches the feasible region on at least point. That solution is the one below:



This profit line touches the feasible region where $b = 195$ and $p = 0$, giving a profit of $P = 30(195) + 40(0) = 5850$. Notice that this is slightly larger than the profit that would be made by completely utilizing all staffing at $b = 120$, $p = 50$, where the profit would be $5600 or at $b = 0$ which would be $p = 110$ and definitely better that (0,0) because that would be $0.

5.7 Limitations of Game Theory

- The biggest issue with game theory is that, like most other economic models, it relies on the assumption that people are rational actors who are self-interested and utility-maximizing. Of course, we are social beings who do cooperate often at our own expense.
- Game theory cannot account for the fact that in some situations we may fall into a Nash equilibrium, and other times not, depending on the social context and who the players are.
- In addition, game theory often struggles to factor in human elements such as loyalty, honesty, or empathy. Though statistical and mathematical computations can dictate what a best course of action should be, humans may not take this course due to incalculable and complex scenarios of self-sacrifice or manipulation.
- Game theory may analyze a set of behaviors but it cannot truly forecast the human element.

Let Us Sum Up

Minimax theory is primarily used in zero-sum games, where one player's gain is exactly balanced by another player's loss. The goal in minimax theory is for each player to minimize their maximum possible loss. This approach involves anticipating the worst-case scenario from the opponent's perspective and selecting a strategy that minimizes the maximum possible loss that could occur. Maximin theory, on the other hand, is focused on decision-making in conditions of uncertainty or risk where outcomes are not necessarily zero-sum. In maximin theory, the decision-maker seeks to maximize their minimum possible payoff or outcome. This strategy is often applied in decision-making scenarios where the worst-case scenario is a critical consideration.

5.2 Check Your Progress

1.What is the primary objective of Minimax theory in game theory?

- A) Maximizing the minimum payoff
- B) Minimizing the maximum payoff
- C) Minimizing the average payoff
- D) Maximizing the total payoff

2.In which type of games is Minimax theory commonly applied?

- A) Cooperative games
- B) Non-zero-sum games
- C) Competitive zero-sum games
- D) Simultaneous games

3.What does the maximin strategy focus on?

- A) Maximizing the maximum possible payoff
- B) Minimizing the maximum possible loss
- C) Minimizing the average payoff
- D) Maximizing the average payoff

4.Which criterion is associated with Maximin theory?

- A) Maximizing the average payoff
- B) Minimizing the minimum payoff
- C) Maximizing the maximum payoff
- D) Minimizing the maximum possible loss

5. In game theory, which theory is concerned with anticipating the worst-case scenario and minimizing potential losses?

- A) Maximin theory
- B) Minimax theory
- C) Nash equilibrium
- D) Zero-sum theory

UNIT SUMMARY

Game theory encompasses a variety of models, including cooperative and non-cooperative games, zero-sum and non-zero-sum games, and static and dynamic games. Key concepts in game theory include Nash equilibrium, where players choose strategies that are mutual best responses, and dominant strategies, which are optimal regardless of opponents' choices. Game theory also explores mixed strategies, where players randomize their actions, and concepts like Pareto efficiency and the minimax theorem. Applications of game theory span economics, political science, biology, and computer science, providing insights into competitive and cooperative behaviors in diverse contexts, from market competition to evolutionary biology and international relations.

Glossary

Nash Equilibrium

A situation in a game where no player can benefit by changing their strategy while the other players keep their strategies unchanged. It represents a state of mutual best responses.

Dominant Strategy

A strategy that is the best for a player, no matter what strategies other players choose. If a player has a dominant strategy, they will always prefer it over any other strategy.

Payoff Matrix

A table that describes the payoffs for each player in a game for every possible combination of strategies. Each cell in the matrix represents the outcome (payoffs) resulting from a specific set of strategies chosen by the players.

Zero-Sum Game

A type of game where one player's gain is exactly balanced by the losses of other players. The total payoff for all players in a zero-sum game is zero.

Cooperative Game

A game where players can form coalitions and make binding agreements to coordinate their strategies and share the resulting payoffs. The focus is on what groups of players can achieve collectively.

Non-Cooperative Game

A game where players make decisions independently and cannot form binding agreements. The focus is on individual strategies and outcomes.

Mixed Strategy

A strategy in which a player chooses among possible actions according to a probability distribution. Unlike a pure strategy, where a player makes a specific choice, a mixed strategy involves randomizing over multiple actions.

Pareto Efficiency

A situation in which it is impossible to make any player better off without making at least one other player worse off. An outcome is Pareto efficient if no player can be improved without hurting another.

Minimax Theorem

In zero-sum games, the minimax theorem states that the player can minimize their maximum possible loss by choosing a specific strategy. The theorem guarantees the existence of an optimal mixed strategy for both players.

Subgame Perfect Equilibrium

A refinement of Nash equilibrium applicable to dynamic games with a sequential structure. A subgame perfect equilibrium requires that players' strategies constitute a Nash equilibrium in every subgame of the original game, ensuring consistent optimal behavior throughout the game.

Check Your Progress- Answers

4.1

B) A strategy where each player's strategy is the best response to the others'
C) B) Prisoner's Dilemma
D) B) A strategy that is the best response regardless of what the other player does
E) D) A game where one player's gain is equal to another player's loss
F) B) Prisoner's Dilemma

4.2

G) B) Minimizing the maximum payoff
H) C) Competitive zero-sum games
I) B) Minimizing the maximum possible loss
J) D) Minimizing the maximum possible loss
K) A) Maximin theory

Self-Assessment Questions

1. Solve the game with the following pay-off matrix:

Player B

		I	II	III	IV
Player A	1	4	2	3	6
	2	3	4	7	5
	3	6	3	5	4

2. Explain the terms i) Rectangular games ii) type of Strategies
3. Solve the following game graphically where pay off matrix for player A has been prepared

1	5	-7	4	2
2	4	9	-3	1

4. Explain the terms Maxmin criteria and Minimax criteria ii) Strategies: Pure and mixed strategies.
5. Solve the following game graphically

	Player B		
Player A	B_1	B_2	B_3
A_1	1	3	11
A_2	8	5	2

6. What are characteristics of a game?

EXERCISE

7. Reduce the following Game by dominance and the fit the game value

PlayerA					
		I	II	III	IV
	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

8. Obtain the optimal strategies for both pensions and the value of the game for two persons zero sum game whose payoff matrix is as follows.

		player-B	
		B1	B2
Player-A	A1	1	-3
	A2	3	5
	A3	-1	6
	A4	4	1
	A5	2	2
	A6	-5	0

9. Explain pay of matrix and types of strategy in game theory?

10. Two criminals are arrested and interrogated separately. They have the option to either cooperate with each other and stay silent or defect and betray the other. The payoffs are as follows:

- If both stay silent, they each get 1 year in prison.
- If one betrays and the other stays silent, the betrayer goes free while the silent one gets 3 years in prison.
- If both betray, they each get 2 years in prison.

What is the Nash equilibrium of this game?

11. Two criminals are arrested and interrogated separately. They have the option to either cooperate with each other and stay silent or defect and betray the other. The payoffs are as follows:

- If both stay silent, they each get 1 year in prison.
- If one betrays and the other stays silent, the betrayer goes free while the silent one gets 3 years in prison.
- If both betray, they each get 2 years in prison.

What is the Nash equilibrium of this game?

12. Two players, A and B, play a game where each player has a penny. They simultaneously choose to show either Heads or Tails. If the pennies match (both Heads or both Tails), Player A wins and keeps both pennies. If the pennies do not match, Player B wins and keeps both pennies. What is the mixed strategy Nash equilibrium of this game?

13. Two hunters can either hunt a stag or a hare. Hunting a stag requires both hunters to cooperate, and they will each get a large payoff. Hunting a hare can be done individually, and it gives a smaller payoff. The payoffs are as follows: If both hunt the stag, they each get 4.

- If one hunts the stag and the other hunts the hare, the stag hunter gets 0, and the hare hunter gets 3.
- If both hunt the hare, they each get 2.

What are the Nash equilibria of this game?

14. Two players must choose the same strategy to achieve a successful outcome. If they coordinate on the same strategy, they receive a payoff. If they choose different strategies, they receive no payoff. The payoffs are as follows:

If both choose A, they each get 3.

If both choose B, they each get 2.

If one chooses A and the other chooses B, they both get 0.

What are the Nash equilibria of this game?

15. In the classic game of Rock-Paper-Scissors, two players simultaneously choose one of the three options. The payoffs are:

Rock beats Scissors (+1 for the player choosing Rock, -1 for the player choosing Scissors).

Scissors beat Paper (+1 for the player choosing Scissors, -1 for the player choosing Paper).

Paper beats Rock (+1 for the player choosing Paper, -1 for the player choosing Rock).

If both players choose the same option, the payoff is 0 for both.

What is the mixed strategy Nash equilibrium of this game?

16. Two players, A and B, each have a penny and can choose to show either Heads or Tails. The payoffs are:

If both show the same side (Heads or Tails), Player A wins (+1) and Player B loses (-1).

If they show different sides, Player B wins (+1) and Player A loses (-1).

Question: What is the mixed strategy Nash equilibrium of this game?

17. Two players, A and B, bet on the outcome of a coin toss. Player A bets on Heads and Player B bets on Tails. The payoffs are:

If the coin lands on Heads, Player A wins $1 and Player B loses $1.

If the coin lands on Tails, Player B wins $1 and Player A loses $1.

Question: What is the expected payoff for each player if the coin is fair?

18. Find Solution of game theory problem using saddle point

Player A\Player B	B1	B2	B3	B4
A1	20	15	12	35
A2	25	14	8	10
A3	40	2	10	5
A4	-5	4	11	0

19. Payoff matrix be formed. Consider the below problem.

		Player B 1	Player B 2
Player A	1	6	-7
	2	1	3
	3	3	1
	4	5	-1

	B_1	B_2	B_3
A_1	2	7	6
A_2	3	4	5
A_3	6	3	8

Identify the saddle point and the optimal strategies for both players.

20. Find Solution of game theory problem using arithmetic method

Player A \ Player B	B1	B2	B3
A1	10	5	-2
A2	13	12	15
A3	16	14	10

21. Consider the example of two criminals arrested for a crime. Prosecutors have no hard evidence to convict them. However, to gain a confession, officials remove the prisoners from their solitary cells and question each one in separate chambers. Neither prisoner has the means to communicate with the other. Officials present four deals, often displayed as a 2 x 2 box.

1. If both confess, they will each receive a three-year prison sentence.
2. If Prisoner 1 confesses, but Prisoner 2 does not, Prisoner 1 will get one year and Prisoner 2 will get five years.
3. If Prisoner 2 confesses, but Prisoner 1 does not, Prisoner 1 will get five years, and Prisoner 2 will get one year.
4. If neither confesses, each will serve two years in prison.

Suggested Readings

1. Introduction to Operations Research" by Frederick S. Hillier and Gerald J. Lieberman
2. Operations Research: An Introduction" by Taha H.A.
3. Operations Research: Applications and Algorithms" by Wayne L. Winston
4. Network Flows: Theory, Algorithms, and Applications" by Ravindra K. Ahuja, Thomas L. Magnanti, and James B. Orlin

Open Source-E Content Link

1. https://www.youtube.com/watch?v=_zhtDJX3kOY
2. https://www.youtube.com/watch?v=27MpB4nd3EI
3. https://www.youtube.com/watch?v=eKSCejfnsmk
4. https://www.youtube.com/watch?v=M8POtpPtQZc
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